1. 

\[ a_3 = a_2 + a_1 + a_0 = 2 + 1 + 2(\frac{1}{2}) = 4 = 2^2 \]
\[ a_4 = a_3 + a_2 + 2a_1 = 4 + 2 + 2 = 8 = 2^3 \]  
we guess \( a_n = 2^{n-1} \) for \( n \geq 0 \)
\[ a_5 = a_4 + a_3 + 2a_2 = 8 + 4 + 2(2) = 16 = 2^4 \]

Proof by strong induction

**Base** \( n = 0 \) \( a_0 = 2^{0-1} = 2^{-1} = \frac{1}{2} = a_0 \)

**Induction Hypothesis** Assume that for some integer \( k \geq 0 \) and for all integers \( t \) such that \( 0 \leq t \leq k \) we have \( a_t = 2^{t-1} \).

We need to prove \( a_{k+1} = 2^k \).

**Proof**

If \( k \leq 2 \), \( k = 0, k = 1, k = 2 \) we already showed that \( a_{k+1} = 2^k \).

If \( k \geq 3 \),

\[ a_{k+1} = a_k + a_{k-1} + 2a_{k-2} \]

and by Induction Hypothesis,

\[ a_k = 2^{k-1}, a_{k-1} = 2^{k-2}, a_{k-2} = 2^{k-3} \]

so

\[ a_{k+1} = 2^{k-1} + 2^{k-2} + 2 \cdot 2^{k-3} = 2^{k-3} (4 + 2 + 2) = 2^{k-3} (8) = 2^{k-3} 2^3 = 2^{k-3+3} = 2^k \]

So by the Principle of Mathematical Induction

\( a_n = 2^{n-1} \) for all \( n \geq 0 \).

2. Since \( 30 = 2 \cdot 3 \cdot 5 \) and \( 180 = 2^2 \cdot 3^2 \cdot 5 \),

if \( x = 2^a \cdot 3^b \cdot 5^c \), and \( y = 2^d \cdot 3^e \cdot 5^f \),

then

\[ \min (a_1, b_1) = 1 \]
\[ \max (a_1, b_1) = 2 \]

\[ \min (a_2, b_2) = 1 \]
\[ \max (a_2, b_2) = 2 \]

\[ \min (a_3, b_3) = 1 \]
\[ \max (a_3, b_3) = 1 \]

so

\( a_3 = b_3 = 1 \)
\( a_1 = 1 \)
\( a_2 = 2 \)
\( b_1 = 2 \)
\( b_2 = 2 \)

we get 4 possibilities \( a_1 = 1, b_1 = 2, a_2 = 1, b_2 = 2, a_3 = 1, b_3 = 1 \)
\(a_1 = 1, \quad b_1 = 2; \quad a_2 = 2, \quad b_2 = 1; \quad a_3 = b_3 = 1\)
\(a_1 = 2, \quad b_1 = 1; \quad a_2 = 1, \quad b_2 = 2; \quad a_3 = b_3 = 1\)
\(a_1 = 2, \quad b_1 = 1; \quad a_2 = 2, \quad b_2 = 1; \quad a_3 = b_3 = 1\)

We get 4 pairs for \((x, y)\):
\[x = 2^1 3^1 5, \quad y = 2^2 3^1 5; \quad (x, y) = (90, 60)\]
\[x = 2^1 3^1 5, \quad y = 2^2 3^1 5; \quad (x, y) = (30, 180)\]
\[x = 2^2 3^2 5, \quad y = 2^3 3^1 5; \quad (x, y) = (60, 90)\]
\[x = 2^3 3^2 5, \quad y = 2^3 3^1 5; \quad (x, y) = (180, 30)\]

If we disregard the order there are only two solutions: \((30, 180)\) and \((60, 90)\).

3. Basis 
\[f_3 = 2 > \alpha = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \frac{1 + \sqrt{5}}{2} < \frac{1 + \sqrt{9}}{2} = 2\]

Induction Hypothesis: Assume that for some integer \(k \geq 3\) and for all integers \(3 \leq l \leq k\) we have \(f_l > \alpha^{l-2}\). We need to prove \(f_{k+1} > \alpha^{k-1}\).

Proof: Since \(k \geq 3\) we also have \(k+1 > 2\).

So \(f_{k+1} = f_k + f_{k-1}\).

If \(k = 3\), then the induction hypothesis does not cover \(f_2\). So \(f_4 = 3 > \alpha^2 = \left(\frac{1 + \sqrt{5}}{2}\right)^2 = \frac{3 + \sqrt{5}}{2}\).

Since \(\frac{3 + \sqrt{5}}{2} < \frac{3 + \sqrt{9}}{2} = 3\),
we can take \(k \geq 4\) so \(k-1 \geq 3\).
And by Induction Hypothesis \(f_k > \alpha^{k-2}\), \(f_{k-1} > \alpha^{k-3}\).

So \(f_{k+1} = f_k + f_{k-1} > \alpha^{k-2} + \alpha^{k-3}\).

\[f_{k+1} > \alpha^{k-3}(1 + \alpha)\]

But \(1 + \alpha = \alpha^2\) so \(f_{k+1} < \alpha^{k-3}, \quad \alpha^2 = \alpha^k\).
4. Basis \( n=1 \): odd and \( f_2 = 1 \) odd

**Induction Hypothesis**: Assume that for some integer \( k \geq 0 \), we have \( f_{3k} \) even, \( f_{3k+1} \) odd, and \( f_{3k+2} \) odd.

We need to prove:

\[
f_{3k+3} \text{ even}, \quad f_{3k+4} \text{ odd}, \quad f_{3k+5} \text{ odd}
\]

**Proof**:

\[
f_{3k+3} = f_{3k+2} + f_{3k+1} = \text{odd} + \text{odd} = \text{even}
\]

\[
f_{3k+4} = f_{3k+3} + f_{3k+2} = \text{even} + \text{odd} = \text{odd}
\]

\[
f_{3k+5} = f_{3k+4} + f_{3k+3} = \text{odd} + \text{even} = \text{odd}
\]

This shows that for all \( n \):

\( f_{3n} \) even, \( f_{3n+1} \) odd, \( f_{3n+2} \) odd.

So \( f_n \) even if and only if \( 3 | n \).

5. Basis \( n=1 \): \( \sum_{j=0}^{0} \binom{1-j}{j} = \binom{1-0}{0} = \binom{1}{0} = 1 = f_2 \)

**Induction Hypothesis**: Assume that for some \( k \geq 1 \), and for all \( 1 \leq l \leq k \),

\[
\sum_{j=0}^{l} \binom{-l-j}{j} = f_{l+1}
\]

We want to prove:

\[
\sum_{j=0}^{\frac{k+1}{2}} \binom{k+1-j}{j} = f_{k+2}
\]

**Proof**:

Let \( S_n = \sum_{j=0}^{\frac{n}{2}} \binom{n-j}{j} \).

I will first illustrate the proof for the case \( k=4, \quad k+1=5 \):

\[
S_5 = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} = \binom{4}{0} + [\binom{3}{0} + \binom{3}{1}] + [\binom{2}{0} + \binom{2}{1}]
\]

\[
= \binom{4}{0} + \binom{3}{1} + \binom{2}{0} + \binom{3}{0} + \binom{2}{1} = S_4 + S_3 = f_5 + f_4 = f_6
\]
\[ S_{k+1} = \binom{k+1}{0} + \binom{k}{1} + \binom{k-1}{2} + \cdots \]

\[ = \binom{k}{0} + \binom{k-1}{0} + \binom{k-1}{1} + \binom{k-2}{1} + \binom{k-2}{2} + \cdots \]

by repeated use of Pascal’s identity on binomial coefficients.

\[ S_k = \binom{k}{0} + \binom{k-1}{1} + \binom{k-2}{2} + \cdots + \binom{k-1}{k-1} + \binom{k-2}{k-2} + \cdots \]

\[ = S_k + S_{k-1} \]

\[ = S_{k+1} + S_k \quad \text{by induction hypothesis} \]

\[ = S_{k+2} \]