Math 437, Homework 5

1. Let \( f(x) = \frac{1}{5 + x} \). Take \( x_0 = 1, x_1 = 2, x_2 = 3, x_3 = 4 \),

(a) Find the Lagrange form, the Newton form and the standard form of the interpolating polynomial \( L_3 \). Check your answer by verifying that \( L_3 \) correctly interpolates \( f \) at the given points.

(b) Find an upper bound for the maximum error
\[
\|f - L_3\|_\infty = \max_{1 \leq x \leq 4} |f(x) - L_3(x)|.
\]

2. (a) Let \( p \) be a polynomial of degree \( n \) and \( L_n \) be the Lagrange interpolation polynomial that interpolates \( p \) at \( x_0 < x_1 < \ldots < x_n \), namely
\[
L(x_i) = p(x_i), \quad i = 0, \ldots, n.
\]
Show that \( L_n(x) = p(x) \) for all \( x \).

(b) Let \( L_n \) be the Lagrange interpolation polynomial that interpolates a function \( f \) at \( x_0 < x_1 < \ldots < x_n \). Show that
\[
f(x) - L_n(x) = \sum_{i=0}^{n} [f(x) - f(x_i)] \ell_i(x)
\]

3. (a) Show that if \( f \) is a polynomial of degree \( k \), then for \( n > k \)
\[
f[x_0, x_1, \ldots, x_n] = 0.
\]

(b) Let \( x_0, x_1, x_2 \) be 3 distinct points. Find the standard form of the polynomial
\[
p(x) = 4 \ell_0(x) + 4 \ell_1(x) + 4 \ell_2(x).
\]
Solve this problem two ways: first by direct computation, second by applying the theorem which says that there is a unique polynomial of degree \( \leq n \) which interpolates a given function at \( n + 1 \) distinct points.

4. Write a program to perform polynomial interpolation at the uniform points and the Chebyshev points on the interval \([-1, 1]\) for the functions
\[
f_1(x) = |x|, \quad f_2(x) = \text{sign}(x).
\]
\((\text{sign}(x) = 1 \text{ if } x > 0, \text{sign}(x) = 0 \text{ if } x = 0, \text{ and } \text{sign}(x) = -1 \text{ if } x < 0)\) Investigate the convergence of \( L_n \) to \( f \) by running the program for different values of \( n \). In the write up include plots of \( f \) and \( L_n \) for \( n = 8, 16, 32 \) for both sets of points. Answer the following questions.

Does \( L_n \) converge pointwise to \( f \) on \([-1, 1]\)?

Does \( L_n \) converge uniformly to \( f \) on \([-1, 1]\)?