Math 437, Homework 3

1. (a) Consider the matrix
\[
A = \begin{pmatrix}
1 & 2 & 5 \\
-1 & 2 & -4 \\
-8 & -1 & 2
\end{pmatrix}.
\]
Compute \(\|A\|_\infty\) and find a vector \(x\) such that \(\|A\|_\infty = \|Ax\|_\infty / \|x\|_\infty\).

(b) Show that if a square matrix \(A\) satisfies an inequality \(\|Ax\| \geq \theta \|x\|\) for all \(x\) with \(\theta > 0\),
then \(A\) is nonsingular and \(\|A^{-1}\| \leq \theta^{-1}\). This is valid for any vector norm and its subordinate matrix norm.

2. (a) Prove that
\[n^{-1}\|A\|_2 \leq n^{-1/2}\|A\|_\infty \leq \|A\|_2 \leq n^{1/2}\|A\|_1 \leq n\|A\|_2\]

(b) Let \(S\) be a real and nonsingular matrix, and let \(\|\cdot\|\) be any norm on \(\mathbb{R}^n\). Define \(\|\cdot\|'\) by \(\|x\|' = \|Sx\|\). Show that \(\|\cdot\|'\) is also a norm on \(\mathbb{R}^n\).

3. Prove that the \(\|\cdot\|_1\) matrix norm can be computed by
\[
\|A\|_1 := \max_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_j \sum_{i=1}^n |a_{ij}|.
\]

4. (a) Show that the eigenvalues of a Hermitian matrix are real.

(b) Prove that if \(A\) is nonsingular and if \(|\lambda| < \|A^{-1}\|^{-1}\), then \(\lambda\) is not an eigenvalue of \(A\).

(c) Show that if there is a polynomial \(p\) without constant term such that
\[
\|I - p(A)\| < 1
\]
then \(A\) is invertible. Find a formula for \(A^{-1}\).

5. Use the Gramm-Schmidt procedure to calculate \(L_1\), \(L_2\), and \(L_3\), where \(\{L_0, L_1, L_2, L_3\}\) is an orthogonal set of polynomials on \((0, \infty)\) with respect to the weight function \(w(x) = e^{-x}\), and \(L_0(x) \equiv 1\). The polynomials obtained from this procedure are called the Laguerre polynomials.