The local torque calculation II

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The asymptotic expansion of the second term in $\tilde{T}$ arising from the radial Green’s function takes the form

$$\tilde{T}_2 \sim \frac{2}{\pi \alpha} A \int_0^\infty \kappa \, d\kappa \, \frac{J_0(\kappa \delta) e^{2\nu(\eta_a - \eta_\rho)}}{\nu \sqrt{1 + z(\rho)^2}} [1 + \cdots]$$

where

$$A = \sum_{m=1}^\infty \sin(\nu \theta) \sin(\nu \theta')$$

$$\nu = \frac{m \pi}{\alpha}$$

$$\kappa^2 = \omega^2 + k^2.$$
Stepping back several steps, we started with

\[ \bar{T} = -\frac{2}{\pi^2 \alpha} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk \ e^{i\omega t + ikz} A \ g_\nu(\rho, \rho') \]  

(12)

where

\[ g_\nu(\rho, \rho') = l_\nu(\kappa \rho_<)K_\nu(\kappa \rho_>) - K_\nu(\kappa \rho)K_\nu(\kappa \rho') \frac{l_\nu(\kappa a)}{K_\nu(\kappa a)} \]

and \( A, \kappa, \) and \( \nu \) are as defined before.
Review of $\bar{T}$

Transforming to polar coordinates in the $\omega$-$k$ and $t$-$z$ planes, we define

$$\omega = \kappa \cos(\gamma) \quad t = \delta \cos(\phi)$$

$$k = \kappa \sin(\gamma) \quad z = \delta \sin(\phi),$$

and obtain

$$\bar{T} = -\frac{2}{\pi^2 \alpha} \int_0^\infty d\kappa \kappa \int_0^{2\pi} d\gamma \ e^{i\kappa \delta \cos(\gamma - \phi)} \sum_{m=1}^{\infty} \sin(\nu \theta) \sin(\nu \theta') g_{\nu}(\rho, \rho').$$
Review of $\bar{T}$

If we write the sine functions in terms of exponentials, then (almost) everything within the $m$ summation becomes a product of exponentials, so

$$\sin(\nu \theta) \sin(\nu \theta') = -\frac{1}{4} (e^{i\nu \theta} - e^{-i\nu \theta})(e^{i\nu \theta'} - e^{-i\nu \theta'})$$

$$= -\frac{1}{4} \left( e^{i\nu (\theta+\theta')} - e^{i\nu (\theta-\theta')} - e^{-i\nu (\theta-\theta')} + e^{-i\nu (\theta+\theta')} \right).$$

By the symmetry of this expression, we can look at just the first term and add in the rest at the end.

$$\bar{T}' = -\frac{1}{2\pi^2 \alpha} \int_0^\infty d\kappa \kappa \int_0^{2\pi} d\gamma \sum_{m=1}^\infty e^{i\kappa \delta \cos(\gamma-\phi)+i\nu (\theta+\theta')} g_{\nu}(\rho, \rho').$$
Uniform Asymptotic Expansion

Taking the uniform asymptotic expansions of the modified Bessel functions yields exponential factors as well, namely with

\[ I_\nu(\nu z) \sim \sqrt{\frac{t}{2\pi \nu}} e^{\nu \eta} \left( 1 + \sum_{k=1}^{\infty} \frac{u_k(t)}{\nu^k} \right) \]

\[ K_\nu(\nu z) \sim \sqrt{\frac{\pi t}{2\nu}} e^{-\nu \eta} \left( 1 + \sum_{k=1}^{\infty} \frac{(-1)^k u_k(t)}{\nu^k} \right) \]

we find

\[ \bar{T}_2' \sim \frac{1}{4\pi^2 \alpha} \int_0^\infty d\kappa \kappa \int_0^{2\pi} d\gamma \sum_{m=1}^{\infty} \frac{1}{\nu \sqrt{1 + z(\rho)^2}} \]

\[ \times e^{2\nu(\eta - \eta_\rho) + i\kappa \delta \cos(\gamma - \phi) + i\nu(\theta + \theta')} \left( 1 - \frac{2u_1(t)}{\nu} + \ldots \right). \]
Change of Variables

Next we make a change of variables and pull any extraneous factors outside of the summation. We take

\[ \kappa \rho = z \nu, \quad d\kappa \rho = dz \nu \]

and so

\[
\bar{T}_2' \sim \frac{1}{4\pi^2 \alpha \rho^2} \int_0^\infty \frac{dz \, z}{\sqrt{1 + z^2}} \int_0^{2\pi} d\gamma \\
\sum_{m=1}^\infty \nu \, e^{\nu \left( 2(\eta_a - \eta_\rho) + \frac{i \, z \delta}{\rho} \cos(\gamma - \phi) + i(\theta + \theta') \right)} \left( 1 - \frac{2u_1(t)}{\nu} + \ldots \right)
\]

\[
\sim \frac{1}{4\pi \alpha^2 \rho^2} \int_0^\infty \frac{dz \, z}{\sqrt{1 + z^2}} \int_0^{2\pi} d\gamma \\
\sum_{m=1}^\infty m \left( e^{\frac{2\pi}{\alpha} (\eta_a - \eta_\rho) + \frac{i \pi z \delta}{\rho \alpha} \cos(\gamma - \phi) + \frac{i \pi}{\alpha} (\theta + \theta')} \right)^m \left( 1 - \frac{\alpha \, 2u_1(t)}{\pi \, m} + \ldots \right).
\]
Polylogarithms

For ease of notation, we define a new function

\[ f^{++}(z) = \frac{2\pi}{\alpha}(\eta_a - \eta_\rho) + \frac{i\pi z\delta}{\rho\alpha} \cos(\gamma - \phi) + \frac{i\pi}{\alpha}(\theta + \theta'). \]

We also note that sums involving inverse powers of the summation index are called *polylogarithms* and have the definition

\[ Li_s(z) \overset{\text{def}}{=} \sum_{k=1}^{\infty} \frac{z^k}{k^s}. \]

These simplify our earlier expression to

\[ T'_2 \sim \frac{1}{4\pi\alpha^2\rho^2} \int_0^\infty \frac{dz}{\sqrt{1 + z^2}} \int_0^{2\pi} d\gamma \left[ Li_{-1}(e^{f^{++}(z)}) - \frac{2\alpha}{\pi} Li_0(e^{f^{++}(z)}) u_1(t) + \cdots \right] \]
Polylogarithms

Some are known in closed form,

\[ \text{Li}_{-1}(e^u) = \frac{e^u}{(1 - e^u)^2} \]
\[ \text{Li}_0(e^u) = \frac{e^u}{1 - e^u} \]
\[ \text{Li}_1(e^u) = -\ln(1 - z), \]

so

\[ T'_2 \sim \frac{1}{4\pi\alpha^2\rho^2} \int_0^\infty \frac{dz}{\sqrt{1 + z^2}} \int_0^{2\pi} d\gamma \left( \frac{e^{f^{++}(z)}}{(1 - e^{f^{++}(z)})^2} - \frac{2\alpha}{\pi} \frac{e^{f^{++}(z)}}{1 - e^{f^{++}(z)}} u_1(t) + \cdots \right). \]
What’s Next

- Find the divergent and finite parts of $\tilde{T}_2$.
- Take second partial derivatives of $\tilde{T}$ and evaluate similarly to find components of the stress tensor.
- Construct $\tilde{T}$ similarly near corners.