Energy density for a massive scalar field in $(1+1)D$

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Introduction

Vacuum energy density for a massive scalar field

This paper is based on Patrick Hays’s paper on a confined massive field in two dimensions. In the paper “Vacuum fluctuations of a confined massive field in two dimensions,” the zero-point energy of a massive scalar field confined to a two-dimensional M.I.T. bag model, is computed.

Motivation

We follow the mathematical style of Fulling’s paper “Vacuum energy as spectral geometry.” The vacuum energy is treated as a purely mathematical problem, an underdeveloped aspect of the spectral theory of self-adjoint second-order differential operators. What I am basically doing is to note the common generalization between the P. Hay’s paper and S. Fulling’s paper.
Vacuum energy density

Boundary vacuum energy from closed and periodic orbits

We consider a finite interval with either a Dirichlet or a Neumann boundary condition at each end. Thus $H = -\frac{d^2}{dx^2} + m^2$ acts in $L^2(0, L)$ on the domain defined by

$$u^{(1-l)}(0) = 0, \quad u^{(1-r)}(L) = 0, \quad l, r \in \{0, 1\}$$

(1)

The Green function can be constructed from $G_\infty$ by the method of images. The Green function can be expressed as

$$G(\omega^2, x, y) = G_\infty(y) + (-1)^l G_\infty(-y) + (-1)^r G_\infty(2L - y) + (-1)^{l+r} G_\infty(2L + y) + (-1)^l G_\infty(-2L + y) + (-1)^{l+r} G_\infty(-2L - y) + (-1)^{l+2r} G_\infty(4L - y) + (-1)^{2l+2r} G_\infty(4L + y) + \cdots$$

(2) (3) (4)

and the above Green function has to satisfy the equation:

$$\delta(x - y) = -\frac{d^2 G}{dx^2} + (m^2 - \lambda)G$$

(5)

This is the same as the equation satified by $G_\infty$ and $G$ in [2] except that $-\lambda$ has been replaced by $m^2 - \lambda$. So, we should be able to use the same formulas as in [2] but we need to replace $\omega(\equiv \sqrt{\lambda})$ by

$$\kappa \equiv \sqrt{\omega^2 - m^2}.$$  

(6)
Green Function

Let’s check from first principles that the new $G_\infty$ satisfies the right Green equations. We want

$$-\frac{\partial^2 G}{\partial x^2} - \kappa^2 G = \delta(x - y),$$

(7)

so for $x - y$ we want

$$\frac{\partial^2 G}{\partial x^2} = -\kappa^2 G.$$  

(8)

Thus

$$G(x, y) = \begin{cases} \text{Ae}^{-i\kappa(x-y)}, & x < y, \\ \text{Be}^{i\kappa(x-y)}, & x > y. \end{cases}$$

(9)

Therefore, our $G_\infty$ is given by

$$G_\infty(\omega^2, x, y) = \frac{i}{2\kappa} e^{-\kappa|x-y|}.$$  

(10)

When we go to the variable $\kappa$ the situation is slightly more complicated; $\kappa$ is not just $\omega$ minus a constant.

Remark: The Weyl and periodic terms will not be the same as in the massless case,
The Hamiltonian, now contains a potential term which comes from the massive scalar field. I will adopt the convention that
\[
(H_x - \kappa) G(\omega^2, x, y) = \delta(x - y). \tag{11}
\]
All we know is that \( G(\omega^2, x, y) \) must satisfy the above equation. Let us not lose sight of our objective, to find the local spectral density from closed and periodic orbits. So, we have
\[
\frac{\kappa}{\omega} \sigma(\omega) \equiv 2\kappa \Im G(\omega^2, x, x)
\]
\[
= \sum_{n=0}^{\infty} (-1)^{n(l+r)} \cos(2\kappa n L) + \sum_{n=0}^{\infty} (-1)^{l+n(l+r)} \cos(2\kappa n L + x) \tag{13}
\]
\[
+ \sum_{n=1}^{\infty} (-1)^{-l+n(l+r)} \cos(2\kappa (nL - x)) + \sum_{n=1}^{\infty} (-1)^{n(l+r)} \cos(2\kappa n L) \tag{14}
\]
\[
= 1 + 2 \sum_{n=1}^{\infty} (-1)^{n(l+r)} \cos(2\kappa n L) + \sum_{n=-\infty}^{\infty} (-1)^{l+n(l+r)} \cos(2\kappa (x + nL)) \tag{15}
\]
\[
\equiv \frac{\kappa}{\omega} (\sigma_{av} + \sigma_{per} + \sigma_{bdry}) \equiv \frac{\kappa}{\omega} (\sigma_{av} + \sigma_{osc}) \tag{16}
\]
where \( \kappa \equiv \sqrt{\omega^2 - m^2} \).
In the case $\xi = \frac{1}{4}$, the contribution of the space derivatives is identical to that of the time derivatives, so we can write
\[
T_{00}(t, x) \equiv E(t, x) = -\frac{1}{2} \frac{\partial}{\partial t} \int_{0}^{\infty} \sigma(\omega) e^{-\omega t} d\omega \equiv E_{\text{Weyl}}(t, x) + E_{\text{per}}(t, x) + E_{\text{bdry}}(t, x). \quad (17)
\]

Using Equation 3.914.1 from [4], we obtain the following expression:
\[
\int_{0}^{\infty} e^{-t\sqrt{m^2 + \kappa^2}} \cos(2nL\kappa) d\kappa = \frac{mt}{\sqrt{t^2 + (2nL)^2}} K_1(m\sqrt{t^2 + (2nL)^2}) \quad (18)
\]

Let’s compute the $E_{\text{Weyl}}$ term for the massive case:
\[
E_{\text{Weyl}}(t) = -\frac{1}{2} \frac{d}{dt} \int_{0}^{\infty} \sigma_{\text{Weyl}}(\omega) e^{-\omega t} d\omega \quad (19)
\]
and doing the change of variables, $\omega^2 = \kappa^2 + m^2$, gives
\[
E_{\text{Weyl}}(t) = -\frac{1}{2\pi} \frac{d}{dt} \int_{0}^{\infty} \frac{\sqrt{\kappa^2 + m^2}}{\kappa} \cdot \frac{\kappa}{\sqrt{\kappa^2 + m^2}} e^{-t\sqrt{\kappa^2 + m^2}} d\kappa \quad (20)
\]
\[
= -\frac{1}{2\pi} \frac{d}{dt} \int_{0}^{\infty} e^{-t\sqrt{\kappa^2 + m^2}} d\kappa = -\frac{1}{2\pi} \frac{d}{dt} mK_1(mt). \quad (21)
\]
When \( \nu \) is fixed and \( z \to 0 \),
\[
K_\nu(z) \sim \frac{1}{2} \Gamma(\nu) \left( \frac{1}{2} z \right)^{-\nu} \quad \text{(Re } \nu > 0) \tag{22}
\]
and in our case, \( \nu = 1 \) and hence
\[
K_1(z = mt) \sim \frac{1}{2} \Gamma(1) \left( \frac{1}{2} mt \right)^{-1} = \frac{1}{mt} \tag{23}
\]
Therefore for small \( mt \), our expression becomes
\[
E_{\text{Weyl}}(t) \sim -\frac{1}{2\pi} \frac{d}{dt} mK_1(mt) = -\frac{1}{2\pi} \frac{d}{dt} \left( \frac{1}{mt} \right) = \frac{1}{2\pi t^2} \tag{24}
\]
To put equation 24 into the usual form for renormalization calculations, we need to expand it in power (Laurent) series in \( t \). The leading term will be \( O(t^{-2}) \) and should match the massless case. The Laurent series in \( t \) for equation can be expressed as
\[
E_{\text{Weyl}}(t) \sim -\frac{1}{2\pi} \frac{d}{dt} \left[ \frac{1}{t} + \frac{1}{4} m(mt) \left( 2 \log(mt) + 2\gamma - 1 - 2 \log(2) \right) + O ((mt)^2) \right] \tag{25}
\]
\[
\sim \frac{1}{2\pi t^2} - \frac{m^2}{4\pi} - \frac{1}{8\pi} m^2 (2 \log(mt) + 2\gamma - 1 - 2 \log(2)) + O(t) \tag{26}
\]
\[
\sim \frac{1}{2\pi} \left[ \frac{1}{t^2} - \frac{m^2}{2} \log \left( \frac{mt}{2} \right) - \frac{m^2}{4} (1 + 2\gamma) \right] + O(t) \tag{27}
\]
The periodic term for the massive case is given by

\[
E_{\text{per}}(t) = -\frac{1}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} (-1)^{n(r+r)} \int_0^\infty \frac{\omega}{\kappa} \sigma_{\text{per}}(\omega) e^{-\omega t} d\omega
\]

\[
= -\frac{1}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} (-1)^{n(r+r)} \int_0^\infty e^{-t \sqrt{m^2 + \kappa^2}} \cos(2nL\kappa) d\kappa
\]

\[
= -\frac{1}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} (-1)^{n(r+r)} \frac{mt}{\sqrt{(2nL)^2 + t^2}} K_1(m \sqrt{(2nL)^2 + t^2})
\]

Therefore,

\[
\lim_{m \to 0} E_{\text{per}}(t) \sim -\frac{1}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} \frac{t}{(2nL)^2 + t^2} - \lim_{m \to 0} \frac{1}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} \frac{m^2 t}{4} \left[ (2 \log \left( m \sqrt{(2nL)^2 + t^2} \right) \right.
\]

\[
+ 2\gamma - 1 - 2 \log(2) \left] + O \left( m^2 (4L^2 n^2 + t^2) \right) \right) \right]
\]

\[
\sim -\frac{1}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} \frac{t}{(2nL)^2 + t^2} = -\frac{1}{\pi} \frac{d}{dt} \frac{1}{4} \left( \pi \coth \left( \frac{\pi t}{2L} \right) - \frac{2}{t} \right)
\]

\[
\sim \frac{\pi}{8L^2} \text{csch}^2 \left( \frac{\pi t}{2L} \right) - \frac{1}{2\pi t^2}
\]

and we can clearly see that the above result agrees with [2, p. 15].
The periodic term, $E_{\text{per}}(t)$, will approach a constant value as $t \to 0$:

$$
\lim_{t \to 0} E_{\text{per}}(t) \sim - \lim_{t \to 0} \frac{1}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} \frac{t}{(2nL)^2 + t^2}
- \lim_{t \to 0} \frac{1}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} \frac{m^2 t}{4} \left[ 2 \log \left( m \sqrt{(2nL)^2 + t^2} \right) + 2\gamma - 1 - 2 \log(2) \right] + O(m^2(4L^2n^2 + t^2))$$

At this point, let’s split the periodic terms into the massless contribution and the massive contribution. The massless contribution to the periodic energy will be denoted by $E_{\text{per}}^{m=0}(t)$ and the massive contribution will be denoted by $E_{\text{per}}^{m}(t)$.

So,

$$
\lim_{t \to 0} E_{\text{per}}^{m=0}(t) = \lim_{t \to 0} \left[ - \frac{1}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} \frac{t}{(2nL)^2 + t^2} \right]
= \lim_{t \to 0} \left[ - \frac{1}{\pi} \pi \sum_{n=1}^{\infty} \frac{d}{dt} \frac{t}{(2nL)^2 + t^2} \right]
= \lim_{t \to 0} \left[ - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{4L^2 n^2 - t^2}{(4L^2 n^2 + t^2)^2} \right]
= \lim_{t \to 0} \left[ \frac{\pi \text{csch}^2 \left( \frac{\pi t}{2L} \right)}{16L^2} - \frac{1}{4\pi t^2} \right]
= \lim_{t \to 0} \left( \frac{\pi \text{csch}^2 \left( \frac{\pi t}{2L} \right)}{8L^2} - \frac{1}{2\pi t^2} \right)
= - \frac{\pi}{24L^2}
$$

and this agrees with [2, pg. 16].
Calculation of the term $E_{bdry}(t, x)$

The interesting term is the boundary term, $E_{bdry}(t, x)$, which is given by

$$E_{bdry}(t, x) = \frac{(-1)^l}{2\pi} \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \int_{0}^{\infty} \frac{\omega}{\kappa} \cos(2\kappa(x + nL)) e^{-\omega t} d\omega$$  \hspace{1cm} (41)

$$= \frac{(-1)^l}{2\pi} \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \int_{0}^{\infty} \cos(2\kappa(x + nL)) e^{-t\sqrt{\kappa^2 + m^2}} d\kappa$$  \hspace{1cm} (42)

Now, we do a change of variables so that we can integrate with respect to $\kappa$ instead of $\omega$. After doing the change of variables we obtain

$$E_{bdry}(t, x) = \frac{(-1)^l}{2\pi} \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \int_{0}^{\infty} \cos(2\kappa(x + nL)) e^{-t\sqrt{\kappa^2 + m^2}} d\kappa$$  \hspace{1cm} (43)

$$= \frac{(-1)^l}{2\pi} \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \frac{mt}{\sqrt{(2(x + nL))^2 + t^2}} K_1(m\sqrt{(2(x + nL))^2 + t^2})$$  \hspace{1cm} (44)

since $\omega d\omega = \kappa d\kappa$. 

In the boundary case, things get more complicated because we now have to deal with position, $x$. Then the boundary term, $E_{bdry}(x, t)$ can be expressed as

$$E_{bdry}(x, t) = \frac{(-1)^l}{2\pi} \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \frac{mt}{\sqrt{4(x + nL)^2 + t^2}} K_1 \left( m\sqrt{4(x + nL)^2 + t^2} \right)$$

Let's go back to computing the boundary term. The boundary term can be expressed as

$$E_{bdry}(x, t) = -\frac{(-1)^l}{2\pi} \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \frac{mt}{\sqrt{4(x + nL)^2 + t^2}} K_1 \left( m\sqrt{4(x + nL)^2 + t^2} \right)$$

Using the asymptotic expansion of $K_1(z)$ for small argument yields

$$E_{bdry}(x, t) \sim \frac{(-1)^l}{2\pi} \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \frac{mt}{\sqrt{4(x + nL)^2 + t^2}} \frac{1}{m\sqrt{4(x + nL)^2 + t^2}}$$

$$= -\frac{(-1)^l}{2\pi} \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \frac{t}{4(x + nL)^2 + t^2}$$

and this agrees with the result obtained in [2].
Let’s go back to the massive case. In the massive case, we quickly discover that we can’t obtain an explicit formula for the infinite sum. Then

\[ E_{bdry}(x, t) = -\frac{(-1)^l}{2\pi} \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \left[ \frac{t}{4(Ln+x)^2 + t^2} \right. \]

\[ + \frac{m^2 t}{4} \left( 2 \log \left( m\sqrt{4(Ln+x)^2 + t^2} \right) + 2\gamma - 1 - 2\log(2) \right) \]

\[ + O \left( (m\sqrt{4(Ln+x)^2 + t^2})^2 \right) \]

Let’s assume that \( l + r \) is an odd integer. For the odd case, we have

\[ E_{bdry}(x, t) = -\frac{(-1)^l}{16L^2} \left[ \coth \left( \frac{\pi(t-2ix)}{2L} \right) \csc \left( \frac{\pi(t-2ix)}{2L} \right) \right. \]

\[ + \coth \left( \frac{\pi(t+2ix)}{2L} \right) \csc \left( \frac{\pi(t+2ix)}{2L} \right) \]

\[ + \frac{(-1)^l}{16L} \left( \csc \left( \frac{\pi(t-2ix)}{2L} \right) + \csc \left( \frac{\pi(t+2ix)}{2L} \right) \right) \]

\[ + \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \frac{(-1)^n}{4} m^2 \left( 2 \log \left( m\sqrt{4(Ln+x)^2 + t^2} \right) + 2\gamma - 1 - 2\log(2) \right) \right. \]

\[ + O(m^4(4(Ln+x)^2 + t^2)) \]

or,

\[ E_{bdry}(x, t) \sim \left[ \frac{\cot \left( \frac{\pi x}{L} \right) \csc \left( \frac{\pi x}{L} \right)}{8L^2} - \frac{t^2 (2 \cot^3 \left( \frac{\pi x}{L} \right) + \cot \left( \frac{\pi x}{L} \right)) \csc \left( \frac{\pi x}{L} \right)}{64L^4} \right. \]

\[ + \left. \left[ -\frac{t^2 \pi m^2 \cot \left( \frac{\pi x}{L} \right) \csc \left( \frac{\pi x}{L} \right)}{16L^2} \right. \right. \]

\[ + \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \frac{(-1)^n}{4} m^2 \left( 2 \log \left( m\sqrt{4(Ln+x)^2 + t^2} \right) - 2\gamma - 1 - 2\log(2) \right) \right. \]

\[ + O(m^4(4(Ln+x)^2 + t^2)) \] + O(t^4)
and when \( t = 0 \), we obtain

\[
E_{bdry}(x, 0) = \frac{\cot \left( \frac{\pi x}{L} \right) \csc \left( \frac{\pi x}{L} \right)}{8L^2} - \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{4} m^2 \left( 2 \log (mL) + 2\gamma - 1 \right)
\]  

(62)

For the even case we obtain,

\[
E_{bdry}(x, t) = -\frac{(-1)^l \pi (\cosh \left( \frac{\pi t}{L} \right) \cos \left( \frac{2x}{L} \right) - 1)}{4L^2 \left( \cos \left( \frac{2x}{L} \right) - \cosh \left( \frac{\pi t}{L} \right) \right)^2} - \frac{(-1)^l \pi m^2 t \sinh \left( \frac{\pi t}{L} \right)}{8L \cos \left( \frac{2x}{L} \right) - \cosh \left( \frac{\pi t}{L} \right)}
\]

\[
+ \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \frac{1}{4} m^2 \left( 2 \log \left( m\sqrt{4(Ln + x)^2 + t^2} \right) + 2\gamma - 1 - 2 \log(2) \right) \right.
\]

\[
+ O \left( m^4 (4(Ln + x)^2 + t^2) \right)
\]

(64)

\[
+ O \left( m^4 (4(Ln + x)^2 + t^2) \right)
\]

(65)

Can we match this result against some results of [1] or Appendix B of the predecessor paper by Bender and Hays [3]?

Ignoring the mass terms, we obtain the following expression:

\[
E_{bdry}(x, t) \sim -\frac{(-1)^l \pi (\cosh \left( \frac{\pi t}{L} \right) \cos \left( \frac{2x}{L} \right) - 1)}{4L^2 \left( \cos \left( \frac{2x}{L} \right) - \cosh \left( \frac{\pi t}{L} \right) \right)^2}
\]  

(66)

and now let’s assume that \( t \) is very small and that \( x \) is fixed. Assuming that \( l = 1 \), and using a power series expansion we obtain the following expression:

\[
E_{bdry}(x, t) \sim \frac{\pi \csc^2 \left( \frac{x}{L} \right)}{8L^2} - \frac{t^2 \left( \pi^3 \left( \cos \left( \frac{2x}{L} \right) + 2 \right) \csc^4 \left( \frac{x}{L} \right) \right)}{32L^4} + O(t^4).
\]  

(67)

When \( t = 0 \), we obtain

\[
E_{bdry}(x, 0) \sim \frac{\pi \csc^2 \left( \frac{x}{L} \right)}{8L^2}
\]  

(68)

and the above result agrees with [2].
Then,

\[
E_{bdry}(x, t) \sim \pi \csc^2 \left( \frac{x}{L} \right) \frac{t^2 (\pi^3 \cos \left( \frac{2x}{L} \right) + 2 \csc^4 \left( \frac{x}{L} \right))}{32L^4} + \pi^2 m^2 t^2 \csc^2 \left( \frac{x}{L} \right) \frac{32}{16L^2} \\
+ \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} \left[ \frac{1}{4} \left( 2m^2 \log \left( 2m\sqrt{(Ln+x)^2} \right) + 2\gamma m^2 - m^2 - 2m^2 \log(2) \right) \right] + O(t^4) 
\]  

(69)

and when \( t = 0 \), we have

\[
E_{bdry}(x, 0) \sim \pi \csc^2 \left( \frac{x}{L} \right) + \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} \frac{1}{4} \left( 2m^2 \log \left( 2m\sqrt{(Ln+x)^2} \right) + 2\gamma m^2 - m^2 - 2m^2 \log(2) \right) 
\]  

(72)

When the mass is sufficiently small or equal to 0, the above analysis yields the correct answers time after time. So far, Dr. Fulling and I haven’t spotted any serious errors with the above asymptotic analysis. It seems to me that the above is valid when \( m \to 0 \) because the above answers also seem to agree with Hay’s paper [1].
The boundary term can be expressed as

\[ E_{bdry}(x, t) = -\frac{(-1)^l}{2\pi} \frac{\partial}{\partial t} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \frac{mt}{\sqrt{4(x + nL)^2 + t^2}} K_1(m\sqrt{4(x + nL)^2 + t^2}) \]  \hspace{1cm} (73)

Let’s assume that \( l + r \) is an even integer. Then we integrate the local energy density and we obtain

\[ E_{bdry}(t) = -\frac{(-1)^l}{\pi} \frac{\partial}{\partial t} \sum_{n=0}^{\infty} \int_0^L \frac{mt}{\sqrt{4(x + nL)^2 + t^2}} K_1(m\sqrt{4(x + nL)^2 + t^2}) \, dx \]  \hspace{1cm} (75)

and hence,

\[ E_{bdry}(t) = -\frac{(-1)^l}{\pi} \sum_{n=0}^{\infty} \int_0^L m \left( \frac{(2Ln - t + 2x)(2Ln + t + 2x)K_1\left(m\sqrt{t^2 + 4(Ln + x)^2}\right)}{(4(Ln + x)^2 + t^2)^{3/2}} \right. \]
\[ - \frac{mt^2K_0\left(m\sqrt{t^2 + 4(Ln + x)^2}\right)}{4(Ln + x)^2 + t^2} \, dx \]  \hspace{1cm} (76)

and doing a change of variables \( x' = Ln + x \), we obtain
\[
E_{bdry}(t) = -\frac{(-1)^l}{\pi} \sum_{n=0}^{\infty} \int_{L_n}^{L(n+1)} \frac{m (2x' - t) (t + 2x') K_1(m \sqrt{t^2 + 4x'^2})}{(t^2 + 4x'^2)^{3/2}} - \frac{mt^2 K_0(m \sqrt{t^2 + 4x'^2})}{t^2 + 4x'^2} \] dx' 
\quad (78)

\[
= -\frac{(-1)^l}{\pi} \int_0^\infty \frac{m (2x' - t) (t + 2x') K_1(m \sqrt{t^2 + 4x'^2})}{(t^2 + 4x'^2)^{3/2}} - \frac{mt^2 K_0(m \sqrt{t^2 + 4x'^2})}{t^2 + 4x'^2} \] dx' 
\quad (79)

and doing another change of variables \( u = 4x'^2 + t^2 \), we have

\[
E_{bdry}(t) = -\frac{(-1)^l}{\pi} \int_t^\infty \frac{m (u^2 - 2t^2) K_1(mu)}{u^3} - \frac{mt^2 K_0(mu)}{u^2} \] \frac{u}{2\sqrt{u^2 - t^2}} du 
\quad (80)

\[
= -\frac{(-1)^l}{\pi} \int_t^\infty \frac{m (u^2 - 2t^2) K_1(mu)}{2u^2 \sqrt{u^2 - t^2}} - \frac{mt^2 K_0(mu)}{u \sqrt{u^2 - t^2}} \] du 
\quad (81)

\[
E_{bdry}(t) = -\left[ \frac{1}{4} \pi m^2 t \text{Ei}(-mt) + \frac{1}{4} \pi m (mt \text{Ei}(-mt) + e^{-mt}) \right] 
\quad (82)

and,

\[
\lim_{t \to 0} E_{bdry}(t) = -\left[ \frac{(-1)^l}{\pi} \lim_{t \to 0} \left( \frac{1}{4} \pi m^2 t \text{Ei}(-mt) + \frac{1}{4} \pi m (mt \text{Ei}(-mt) + e^{-mt}) \right) \right] 
\quad (83)
\]

\[
= -\frac{(-1)^l}{\pi} \left( \frac{m \pi}{4} \right) = -\frac{(-1)^l m}{4} 
\quad (84)

And for the Dirichlet case \( l = 0 \) we obtain

\[
E_{bdry}(0) = -\frac{m}{4} 
\quad (85)
Future Work

There are 4 calculations in Section 4 of [2]:

- local spectral density ($\sigma$, p. 12),
- “global” eigenvalue density ($\rho$ or $N$, p. 13 and p. 14),
- total energy ($E$, pp. 15-16), and
- local energy density ($T_{00}$ or $E(t, x)$, pp. 18-20).

Question: Where do we stand on these four calculations?

Answer: And, my answer is simple, I have been focusing all of my attention on the local energy density calculations. I ignored the other three calculations because I thought obtaining the local energy density was the top priority. Hopefully, I will get around to improving the structure of the paper itself, but before I do that I would like to receive more feedback.

The massive analogs of the formulas for $\rho_{\text{Weyl}}, \rho_{\text{per}}, \rho_{\text{bdry}}$ will be computed along the following lines:

- The massive analogs of the formulas for $\sigma(\omega)$ are related to the massless case by simply substituting $\kappa$ for $\omega$ and also multiplying it by the factor $\frac{\pi \omega}{\kappa}$. 
At this point, the research notes lack structure, but the notes don’t lack direction. The direction that I am taking is now to compute the massive analog of the counting function $N(\omega)$.

Once again, there should be agreement between the massive counting function and the massless counting function when $m \to 0$.

Let’s examine the global situation first. The eigenvalue density is

$$
\rho(\omega) = \int_0^L \sigma(\omega, x) \, dx = \rho_{\text{Weyl}}(\omega) + \rho_{\text{per}}(\omega) + \rho_{\text{bdry}}(\omega),
$$

where

$$
\rho_{\text{Weyl}}(\omega) = \int_0^L \sigma_{av} \, dx = \int_0^L \frac{\omega}{\pi \kappa} \, dx = \frac{L \omega}{\pi \kappa},
$$

$$
\rho_{\text{per}} = \frac{2L \omega}{\pi \kappa} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \cos(2\kappa n L),
$$

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$$

$$
\rho_{\text{per}} = \frac{2L \omega}{\pi \kappa} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \cos(2\kappa n L),
$$

$$
\rho_{\text{bdry}} = \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n(l+r)} \omega}{\kappa^2} [\sin(2\kappa L(n+1)) - \sin(2\kappa L n)],
$$

where $\kappa = \sqrt{\omega^2 - m^2}$. 
The eigenvalue counting function $N(\omega)$ is zero for $\omega < m$ and $\int_{0}^{\omega} \rho$ for $\omega > m$. Therefore (for $\omega > m$),

$$N_{\text{Weyl}}(\omega) = \frac{L}{\pi} \int_{0}^{\kappa} \frac{\omega \cdot \kappa}{\kappa} d\kappa = \frac{L\kappa}{\pi} = \frac{L\sqrt{\omega^2 - m^2}}{\pi}.$$  \hspace{1cm} (91)

$$N_{\text{per}}(\omega) = \frac{2L}{\pi} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \int_{0}^{\kappa} \frac{\sqrt{\kappa^2 + m^2}}{\kappa} \cos(2nL\kappa) d\kappa$$

$$= \frac{2L}{\pi} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \frac{\sin(2nL\kappa)}{2Ln}$$

$$= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n(l+r)}}{n} \sin(2nL\kappa)$$ \hspace{1cm} (92)

$$= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n(l+r)}}{n} \sin(2nL\kappa)$$ \hspace{1cm} (93)

The Fourier series in $N_{\text{per}}$ can be evaluated to a sawtooth function. [See GR 1.441.1, GR 1.441.3, and [2] pp. 14 and pg. 9-10].

$$\rho_{\text{bdry}}(\omega) = \int_{0}^{L} \sigma_{\text{bdry}}(\omega) dx = \frac{(-1)^{l}}{\pi} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \int_{0}^{L} \frac{\omega}{\kappa} \cos(2\kappa(x + nL)) dx$$

$$= \frac{(-1)^{l}}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \omega \kappa^2 [\sin(2\kappa L(n + 1)) - \sin(2\kappa nL)]$$ \hspace{1cm} (94)

and,

$$= \frac{(-1)^{l}}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \omega \kappa^2 [\sin(2\kappa L(n + 1)) - \sin(2\kappa nL)]$$ \hspace{1cm} (95)
\[ N_{\text{bdry}}(\omega) = \int_{m}^{\omega} \rho_{\text{bdry}}(\omega) \, d\omega = \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \int_{0}^{\kappa} \frac{\sqrt{\kappa^2 + m^2}}{\kappa^2} \left[ \sin(2\kappa L(n + 1)) - \sin(2\kappa n L) \right] \frac{\kappa}{\sqrt{\kappa^2 + m^2}} \, d\kappa \]

\[ = \frac{(-1)^l}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^{n(l+r)} \int_{0}^{\kappa} \frac{1}{\kappa} \left[ \sin(2\kappa L(n + 1)) - \sin(2\kappa n L) \right] \, d\kappa \]

In other words, we end up with the following expression:

\[ N_{\text{bdry}}(\omega) = \begin{cases} \frac{(-1)^l}{\pi} \sum_{n=0}^{\infty} (-1)^{n(l+r)} \int_{0}^{\kappa} \frac{\sin(2\kappa L(n+1)) - \sin(2\kappa n L)}{\kappa} \, d\kappa & \text{if } l + r \text{ is even}, \\ 0 & \text{if } l + r \text{ is odd}. \end{cases} \] (99)

Consider the regularized vacuum energy

\[ E(t) = -\frac{d}{2} \frac{1}{dt} \int_{0}^{\infty} \rho(\omega) e^{-\omega t} \, d\omega \equiv E_{\text{Weyl}}(t) + E_{\text{per}}(t) + E_{\text{bdry}}(t) \] (100)

where

\[ E_{\text{per}}(t) = -\frac{L}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \int_{0}^{\infty} \cos(2\kappa n L) e^{-t\sqrt{\kappa^2 + m^2}} \, d\kappa \]

\[ = -\frac{L}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \frac{m t}{\sqrt{t^2 + (2nL)^2}} K_1 \left( m\sqrt{t^2 + (2nL)^2} \right) \] (102)
Taking the limit of $m \to 0$ of $E_{\text{per}}(t)$ yields

$$\lim_{m \to 0} E_{\text{per}}(t) = \lim_{m \to 0} -\frac{L}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \frac{mt}{\sqrt{t^2 + (2nL)^2}} K_1 \left( m \sqrt{t + (2nL)^2} \right)$$

$$\sim \lim_{m \to 0} -\frac{L}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \frac{mt}{\sqrt{t^2 + (2nL)^2}}$$

$$\times \left[ \frac{1}{z} + \frac{1}{4} z (2 \log(z) + 2\gamma - 1 - 2 \log(2)) + O(z^2) \right]$$

$$= -\frac{L}{\pi} \frac{d}{dt} \sum_{n=1}^{\infty} (-1)^{n(l+r)} \frac{t}{t^2 + (2nL)^2},$$

where $z = m \sqrt{t^2 + (2nL)^2}$. So, the massive analog of the periodic energy agrees with the massless case.

The End


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