1 Bounded operators & examples

Let $V$ and $W$ be Banach spaces. We say that a linear transformation $L : V \to W$ is bounded if and only if there is a constant $K$ such that $\|Lv\|_W \leq K\|v\|_V$ for all $v \in V$. Equivalently, $L$ is bounded whenever

$$
\|L\|_{op} := \sup_{v \neq 0} \frac{\|Lv\|_W}{\|v\|_V}
$$

is finite. $\|L\|_{op}$ is called the norm of $L$. Frequently, the same operator may map another space $\tilde{V} \to \tilde{W}$, rather than $V \to W$. When this happens, we will need to note which spaces are involved. For instance, if $V$ and $W$ are the spaces involved, we will use the notation $\|L\|_{V \to W}$ for the operator norm. In addition to the expression given in (1.1), it is easy to show that $\|L\|_{op}$ is also given by

$$
\|L\|_{op} := \min \{K > 0 : \|Lv\|_W \leq K\|v\|_V \forall v \in V\}.
$$

As usual, we say $L : V \to W$ is continuous at $v \in V$ if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $\|Lu - Lv\|_W < \varepsilon$ whenever $\|u - v\|_V < \delta$. Of course, this is just the standard definition of continuity. Be aware that it holds whether or not $L$ is linear. When $L$ is linear, the distinction between $u, v$ becomes irrelevant, because $\|Lu - Lv\|_W = \|L(u - v)\|_W$. From this it immediately follows that $L$ will be continuous at every $v \in V$ whenever it is continuous at $v = 0$. The proposition below connects boundedness and continuity for linear transformations. The proof is left as an exercise.

**Proposition 1.** A linear transformation $L : V \to W$ is continuous if and only if it is bounded.

We will now provide a number of examples of bounded operators and unbounded operators.
Example 1. Let \( L : C[0, 1] \to C[0, 1] \) be given by \( Lu(x) = \int_0^1 k(x, y)u(y)dy \), where \( k \in C(R) \), \( R = [0, 1] \times [0, 1] \). We have that \(|Lu(x)| \leq \int_0^1 |k(x, y)| |u(y)|dy\), so \(|Lu(x)| \leq \|k\|_{C(R)}\|u\|_{C([0,1])}\). Consequently, \( \|L\|_{C \to C} \leq \|k\|_{C(R)}\|u\|_{C([0,1])} \).

Example 2. Hilbert-Schmidt operators.

Definition 1. Let \( R = [0, 1] \times [0, 1] \) and let \( k : R \to \mathbb{R} \). If \( k \in L^2(R) \), then \( k \) is called a Hilbert-Schmidt kernel.

Proposition 2. Let \( k \) be a Hilbert-Schmidt kernel. The linear operator \( Lu(x) = \int_0^1 k(x, y)u(y)dy \) maps \( L^2[0, 1] \to L^2[0, 1] \) and is bounded. Moreover, \( \|L\|_{L^2 \to L^2} \leq \|k\|_{L^2(R)} \).

Proof. Since \( k(x, y) \in L^2(R) \), \( \int_R |k(x, y)|^2dxdy < \infty \), we have that \(|k(x, y)|^2 \in L^1(R) \). Fubini’s theorem then implies that \( \int_0^1 |k(x, y)|^2dy \) exists for almost every \( x \) and, in \( x \), is in \( L^1[0, 1] \). But this also implies that for almost every \( x \), \(|k(x, y)|^2 \) is \( L^2 \) in \( y \). Hence, by Schwarz’s inequality,

\[
|Lu(x)|^2 = \left| \int_0^1 k(x, y)u(y)dy \right|^2 \leq \int_0^1 |k(x, y)|^2dy \underbrace{\int_0^1 |u(y)|^2dy}_{\|u\|_{L^2}^2}.
\]

Integrating both sides in \( x \) then yields \( \|Lu\|_{L^2[0,1]}^2 \leq \|k\|_{L^2(R)}^2\|u\|_{L^2[0,1]}^2 \), so \( \|Lu\|_{L^2[0,1]} \leq \|k\|_{L^2(R)}\|u\|_{L^2[0,1]} \). Then by (1.2), we see that \( \|L\|_{L^2 \to L^2} \leq \|k\|_{L^2(R)} \), which completes the proof.

Example 3. Consider \( L^2[0, 1] \). The differentiation operator \( D = \frac{d}{dx} \) is defined on all \( f \in C^1[0, 1] \), which is dense in \( L^2 \) because it contains the set of polynomials. The question is whether \( D \) is bounded, or at least can be extended to a bounded operator on \( L^2 \). The answer is no. Let \( u_n(x) := \sqrt{2} \sin(n \pi x) \). These functions are in \( C^1[0, 1] \) and they satisfy \( \|u_n\|_{L^2} = 1 \). Since \( Du_n = n \pi \sqrt{2} \cos(n \pi x) \), \( \|Du_n\|_{L^2} = n \pi \). Consequently,

\[
\frac{\|Du_n\|_{L^2}}{\|u_n\|_{L^2}} = n \pi \to \infty, \quad \text{as} \; n \to \infty.
\]

Thus \( D \) is an unbounded operator on \( L^2[0, 1] \).

The situation changes if we use a different space. Consider the Sobolev space \( H^1[0, 1] \), which has the inner product

\[
\langle f, g \rangle_{H^1} = \int_0^1 f(x)\overline{g(x)} + f'(x)\overline{g'(x)}dx.
\]
The operator $D : H^1 \to L^2$ turns out to be bounded. In fact, one can show that $\|D\|_{H^1 \to L^2} = 1$. (It’s easy to show that $\|D\|_{H^1 \to L^2}$ is at most 1. Showing that it’s exactly one requires more work.)

2 Closed subspaces

The usual definition of subspace holds for Banach spaces and for Hilbert spaces. Such subspaces inherit norms and/or inner products from the larger spaces. They are said to be closed if they contain all of their limit points.

Finite dimensional subspaces are always closed. Earlier, when we discussed completeness of an orthonormal set $U = \{u_n\}_{n=1}^\infty$ in a Hilbert space $\mathcal{H}$, we saw that the space $H_U = \{f \in \mathcal{H} : f = \sum_n \langle f, u_n \rangle u_n\}$ is closed in $\mathcal{H}$. When $C[0,1]$ is considered to be a subspace of $L^2[0,1]$, it is not closed. However, $C[0,1]$ is a closed subspace of $L^\infty[0,1]$.

Given a subspace $V$ of a Hilbert space $\mathcal{H}$, we define the orthogonal complement of $V$ to be

$$V^\perp := \{ f \in \mathcal{H} : \langle f, g \rangle = 0, \forall g \in V \}.$$

**Proposition 3.** $V^\perp$ is a closed subspace of $\mathcal{H}$.

**Proof.** Let $\{f_n\}_{n=1}^\infty$ be a sequence in $V^\perp$ that converges to a function $f \in \mathcal{H}$. Since each $f_n$ is in $V^\perp$, $\langle f_n, g \rangle = 0$ for every $g \in V$. Also, because the inner product is continuous, $\lim_{n \to \infty} \langle f_n, g \rangle = \langle f, g \rangle$. It immediately follows that $\langle f, g \rangle = 0$, so $f \in V^\perp$. Consequently, $V^\perp$ is closed in $\mathcal{H}$. \hfill \Box

Bounded linear operators mapping $V \to W$, where $V$ and $W$ are Banach spaces, have all of the usual subspaces associated with them. Let $L : V \to W$ be bounded and linear. The domain of $L$ is $D(L) = V$. The range of $L$ is defined as $R(L) := \{ w \in W : \exists v \in V \text{ for which } L v = W \}$. Finally, the null space (or kernel) of $L$ is $N(L) := \{ v \in V : L v = 0 \}$.

**Proposition 4.** If $L : V \to W$ is bounded and linear, then the null space $N(L)$ is a closed subspace of $V$.

**Proof.** The proof again relies on the continuity of $L$. If $\{f_n\}_{n=1}^\infty$ is a sequence in $N(L)$ that converges to $f \in V$. By Proposition 1, $L$ is continuous, so $\lim_{n \to \infty} L f_n = L f$. But, because $f_n \in N(L), L f_n = 0$. Combining this with $\lim_{n \to \infty} L f_n = L f$, we see that $L f = 0$ and so $f \in N(L)$. Thus, $N(L)$ is a closed subspace of $V$. \hfill \Box