1. Solve the initial value problems below.

(a) (10 pts.) \( y' = (4 + y^2) \sin t, \quad y(\pi) = 0 \).

(b) (10 pts.) \( (\cos x)y' - (\sin x)y = x \cos x, \quad y(\pi) = 0 \).

2. (15 pts.) Let \( L[y] = y'' - y' - 6y \). Find \( L[e^{rx}] \), and determine a fundamental set of solutions to \( L[y] = 0 \). Solve the initial value problem \( L[y] = 0, \ y(0) = 1, \) and \( y'(0) = -1 \).

3. (15 pts.) A population model is governed by the logistic equation,

\[
\frac{dP}{dt} = 3P - P^2 \quad \text{where} \ P \geq 0.
\]

Use dfield5 to find the equilibrium points (constant solutions), and to determine whether each is stable or unstable. Can the population increase from 1.5 to 2.2, or from 2.9 to 3.1? As \( t \to +\infty \), does \( P(t) \) approach a limit?

4. Let \( L[y] = x^2 y'' - xy' + y \). You are given that \( y_1(x) \) satisfies \( L[y_1] = 0 \).

(a) (10 pts.) For any function \( u(x) \), find \( L[u y_1] \).

(b) (10 pts.) Use the result above and reduction of order to find a second solution to \( L[y] = 0 \), given that \( y_1(x) = x \).

5. (7 pts.) You are given that \( y_1 = \cos(x) \) and \( y_2 = \sin(\pi/2 - x) \) satisfy \( y'' + y = 0 \). Calculate the Wronskian \( W[y_1, y_2] \) and evaluate it at \( x = 0 \). From your result, show that the trigonometric identity \( \cos(x) = \sin(\pi/2 - x) \) holds.

6. (23 pts.) A 20% alcohol solution flows into a tank at a constant rate of 5 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the same rate. Given that the tank initially held 200 L of a 1% alcohol solution, determine the concentration of alcohol in the tank at any time \( t > 0 \). Use MATLAB to make a plot of the concentration. Either graphically or analytically, determine the time when the concentration reaches 7%. Indicate this point on your plot.