Mid-Term—Math 308-509

Instructions: Show all work on the paper provided. Only scientific calculators are allowed.

PART I

1. (10 pts.) Find the solution to \( y' = (9 - y^2) \sin t, \ y(\pi) = 0 \).

2. (5 pts.) If \( L \) is the operator \( L = D^2 -xD + 3 \), find \( L[e^{2x}] \).

3. (15 pts.) Compute the Wronskian of the set \( \{xe^{-2x}, xe^{-3x}\} \). Is the set linearly dependent or linearly independent? Can \( \{xe^{-2x}, xe^{-3x}\} \) be a fundamental set for \( L[y] = y'' + p(x)y' + q(x)y = 0 \), with \( p \) and \( q \) being continuous for all \( x \)? Explain.

4. (10 pts.) Verify that the equation \( y^2dx + (2xy - \sin(y))dy = 0 \) is exact, and solve the IVP comprising this equation together with \( y(1) = \pi \).

5. (10 pts.) The Gompertz equation, which is given below, is used to model population growth.

\[
\frac{dp}{dt} = -rp \ln(p/K), \quad r > 0, K > 0 \text{ are constants, and } p \geq 0
\]

Find the equilibrium points. For what points \((t_0, p_0)\) are we assured that the associated IVP with initial conditions \( p(t_0) = p_0 \) can always be uniquely solved? At \( p(3) = K/2 \), is the population increasing or decreasing? Can the population go from \( p(3) = K/2 \) to \( p = 2K \)? Explain.

6. (10 pts.) Use reduction of order to find a second linearly independent solution to \( x^2y'' - 5xy' + 9y = 0 \), given that \( y_1 = x^3 \) is a solution.

PART II

1. (20 pts.) The heat in a building is turned on at 6 am and set to \( 68^0 \). At that time the building’s temperature is \( 55^0 \), the outside temperature is \( 45^0 \), and remains so the entire day. If the time constant for the building is 2 hours and if the building’s time constant with heating or cooling is \( 1/3 \) hours, then what is the temperature in the building at 10 am?

2. (20 pts.) A nitric acid solution flows at a constant rate of 6 L/min into a large tank that initially held 200 L of a 0.5% nitric acid solution. The solution inside the tank is kept well stirred and flows out of the tank at a rate of 8 L/min. If the solution entering the tank is 20% nitric acid, determine the volume of nitric acid at any time \( t > 0 \).