Publishers' page

Publishers' page

Publishers' page

Publishers' page

Preface

The main justification for this book is that there have been significant advances in continued fractions over the past decade, but these remain for the most part scattered across the literature, and under the heading of topics from algebraic number theory to theoretical plasma physics.

We now have a better understanding of the rate at which assorted continued fraction or greatest common denominator (gcd) algorithms complete their tasks. The number of steps required to complete a gcd calculation, for instance, has a Gaussian normal distribution.

We know a lot more about *badly approximable* numbers. There are several related threads here. A badly approximable number is a number x such that $\{q|p - qx| \colon\ p, q \in \mathbb{Z} \text{ and } q \neq 0\}$ is bounded below by a positive constant; badly approximable numbers have continued fraction expansions with bounded partial quotients, and so we are led to consider a kind of Cantor set E_M consisting of all $x \in [0, 1]$ such that the partial quotients of x are bounded above by M. The notion of a badly approximable *rational* number has the ring of crank mathematics, but it is quite natural to study the set of rationals r with partial quotients bounded by M. The number of such rationals with denominators up to n, say, turns out to be closely related to the Hausdorff dimension of E_M, (comparable to $n^{2\dim E_M}$) which is in turn related to the spectral radius of linear operators $L_{M,s}$, acting on some suitably chosen space of functions f, and given by $L_{M,s}f(t) = \sum_{k=1}^{m}(k + t)^{-s}f(1/(k + t))$. Similar operators have been studied by, among others, David Ruelle, in connection with theoretical one-dimensional plasmas, and they are related to entropy.

Alongside these developments there has been a dramatic increase in the computational power available to investigators. This has been helpful on the theoretical side, as one is more likely to seek a proof for a result when, following computations and graphical rendering of the output, that result leaps off the screen.

Consider, for instance, the venerable Hurwitz complex continued fraction algorithm. This algorithm takes as input a complex number ξ (say, inside the unit square centered on 0), and returns a sequence $\langle a_n \rangle$ of Gaussian integers $a_1, a_2, \ldots$, all outside the unit disk, such that

$$\xi = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}.$$

The algorithm uses an auxiliary sequence $\langle \xi_n \rangle$, with $\xi_1 = \xi$ and $\xi_{n+1} = 1/\xi_n - a_n$. Is there any particular pattern to the distribution of the ξ_k's? What sorts of numbers have atypical expansions?

These questions are analogs of questions for which, in the case of the real numbers and the classical continued fraction expansion, answers are known or suspected. Almost always, the expansion of a randomly chosen real input $\xi \in (0, 1)$ will have the property that if $\xi = 1/(a_1 + 1/(a_2 + \ldots))$, then the ξ_n given by the same recurrence relation as mentioned above are distributed according to the *Gauss density* $1/((1+x)\log 2)$. Quadratic irrationals have ultimately periodic continued fraction expansions, and therefore, their ξ_n are not so distributed, but in the case of real inputs these seem to be the only algebraic exceptions. Back in the complex case, to assemble some tens of thousands of data points (a bare minimum considering that a 1-megapixel image is hardly high resolution) can require extensive computations. But once this is done, it turns out there are some surprises-there are algebraic numbers with expansions atypical of randomly chosen inputs, yet not of degree 2. This is discussed in chapter 5.

Passing from the complex numbers, at once one-dimensional and two-dimensional, we turn our attention to simultaneous diophantine approximation of real n-tuples $\xi = (\xi_1, \ldots, \xi_n)$. Here, we are looking for a positive integer q, and further integers $(p_1, \ldots, p_n)$, such that $e(q, \xi) := \max\{|p_j - q\xi_j|, 1 \le j \le n\}$ will be 'small'. The Dirichlet principle guarantees that there are infinitely many choices of q such that, in combination with the unique sensible choice of the p_j's, gives $E(q, \xi) \ll q^{-1/n}$. (If ξ contains only rational entries, these q are eventually just multiples of a common-denominator representation of ξ, and the errors are zero.)

Computing good choices of q by head-on search is computationally prohibitively expensive, as the sequence of good q tends to grow exponentially. We discuss two algorithms for this task. Both rely upon the insight that approximation of ξ is related to the task of finding *reduced* bases of $(n+1)\times(n+1)$ lattices of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ -\xi_1 & -\xi_2 & \dots & \epsilon \end{pmatrix}.$$

There are different ways to give exact meaning to the notion of a reduced lattice basis. The general idea is that the vectors should form an integral basis of the lattice, and they should be short.

The Gauss lattice reduction algorithm treats the two-dimensional case, and has recently been analyzed by Daudé, Flajolet, and Vallée. It is discussed in chapter 2.

The Lagarias geodesic multidimensional continued fraction algorithm uses a form of Minkowski reduction, which is computationally feasible for modest dimensions and gives in a sense best-possible answers, while the Lenstra-Lenstra-Lovasz algorithm gives much quicker answers when the dimension is large, but at the risk that the results obtained may not be quite so good. These are discussed in chapter 6.

Some numbers, for instance, e, have continued fraction expansions featuring fairly frequent, ever-larger partial quotients. (Liouville numbers take this to an extreme!) Others, for instance, $\sqrt{2}$, have continued fraction expansions with bounded partial quotients. Chapter 3 is dedicated to this latter type of number, further broken down as the union $\cup_{M=2}^{\infty} E_M$ of continued fraction Cantor sets. Of course, even E_2 is uncountable, and quadratic irrationals are but the tip of this iceberg. The size of the E_M is best understood in the context of Hausdorff dimension, and we discuss this. The *discrepancy* of the sequence $\langle n\alpha\rangle$, as well as the behavior of related sums, is discussed as well. Chapters 4 and 9 also treat the topic of E_M.

In chapter 4, we look at the ergodic theory of continued fractions. (There is a recent book by Dajani and Kraaikamp which treats the topic more extensively.) Portions of this chapter first appeared in *New York J. of Math.* **4**, pp. 249-258.

Chapter 5 is devoted to the complex continued fraction algorithms of Asmus Schmidt and of Adolf Hurwitz. Interest in the former has perhaps suffered from the lack of a convenient algorithms for computer implementation, while it seems not to have been recognized that the latter enjoys many good properties beyond those initially established by Hurwitz. In particular, there is an analog to the Gauss density for the Hurwitz algorithm; it even makes a pretty picture, and is featured on the cover.

Chapter 6 is devoted to multidimensional Diophantine approximation, and in particular, to the so-called *Hermite approximations* to numbers and vectors, and the Lagarias geodesic multidimensional continued fraction algorithm.

Chapter 7 discusses an interesting generalization of the approximation properties of quadratic irrationals. The field $\mathbb{Q}(\sqrt{2})$, seen as a vector space over $\mathbb{Q}$, has the canonical basis $\{1, \sqrt{2}\}$. The field $\mathbb{Q}(2^{1/3})$, seen also as a vector space over $\mathbb{Q}$, has canonical basis $\{1, 2^{1/3}, 2^{2/3}\}$. Thus from a certain point of view, we should expect theorems about quadratic irrationals to have analogues not in the context of rational approximations to a single number such as $2^{1/3}$, but rather in the context of simultaneous diophantine approximation to $(2^{1/3}, 2^{2/3})$. And so it is.

Chapter 8 discusses Marshall Hall's theorem concerning sums of continued fraction Cantor sets. This theorem has undergone various iterations and the current strongest version seems to be due to Astels. We give a taste of his approach.

Chapter 9 discusses the functional-analytic techniques arising out of work by K. I. Babenko, E. Wirsing, D. Ruelle, D. Mayer, and others, or, if one goes back all the way, out of the conjecture by Gauss concerning the frequency of the partial quotients in continued fraction expansions of typical numbers. Combined with modern computing power, it becomes possible to evaluate, say, the Wirsing constant, to many digits. Portions of this chapter first appeared in Number Theory for the Millennium, Vol II, pp. 175-194.

Chapter 10 discusses a dynamical-systems perspective, related to chapter 9 but bringing new tools to the analysis. This approach has scored a real triumph recently, with the result by V. Baladi and B. Vallée that all the standard variants of the Euclidean algorithm have Gaussian normal distribution statistics for a wide variety of measures of the work they must do on typical inputs.

Chapter 11 discusses so-called *conformal iterated function systems*. Much of the material of continued fractions can be seen as an instance of such systems. In this topic, the names Mauldin and Urbański are prominent.

Finally, chapter 12 discusses convergence of continued fractions, in the spirit of Perron's classic book, and the later classic by Jones and Thron.

Little in this chapter is new, but it would be a pity to omit all mention of these wonderful results. One tidbit that emerges from an extensive theory going back to Jacobi and Laplace is a (standard) continued fraction expansion for $e^{2/k}$. Also discussed are analytical continued fraction expansions, such as those for $\log(1+z)$, $\tanh(z)$, and e^z. This also gives us the classical Euler expansion of e itself as a continued fraction, so that at the end of the book we have come full circle to the beginnings of the subject matter.

What's New Quite a bit, actually. Theorems 3.2, theorems 4.2 and 4.3, Theorems 5.1-5.5, Theorem 6.2, Theorems 7.1-7.3, the estimate for the Wirsing constant in chapter 9, Theorem 9.5, the proof of Theorem 12.1, and Theorem 12.5, are, so far as the author is aware, new.

Acknowledgments My thanks first to colleagues who gave advice or encouragement, among them O. Bandtlow, I. Borosh, P. Cohen, H. Diamond, O. Jenkinson, J. C. Lagarias, D. Mauldin, and B. Vallée.

In a sense, it should go without saying that no book is written from scratch. Everything has roots, and the content presented here is a distillation and compilation of the work of hundreds of researchers, over and above the author's own contribution. Many of their names have been left out, but only because citation chains must be pruned, or through an oversight.

It should also go without saying that authors don't work in a vacuum. For creating and explaining LaTeX, (particularly as it applies to books,) my thanks to the Knuths and to George Grätzer, respectively. For editorial assitance and help with LaTeX, my thanks to Mary Chapman and Robin Campbell, respectively.

It should further go without saying that authors are far from disembodied scholarly entities. The kilo-hours that are devoted to preparing a manuscript are a gift from their families. As leaf litter forms the floor of a forest, on occasion, yellow legal pad litter formed a floor of sorts across one room and another. My thanks to Pam for the gifts of time to work, and patience with the worker.

My thanks finally to Lucille, who always held that scholars should write books, but would have had to wait until her 114th birthday or so to see the result. These things should go without saying, but they shouldn't go unsaid.

Contents

Chapter 1

Powers of an Algebraic Integer

1.1 Introduction

Let $\alpha \in \mathbb{R}$ be an algebraic integer of degree n. For $\sigma > 0$ and for integer $q > 0$, we call the triple (σ, q, α) *good* if there exist integers $p_1, p_2, \ldots p_{n-1}$ so that for $1 \leq j \leq n-1$,

$$|p_j - q\alpha^j| \leq \sigma q^{-1/(n-1)}.$$

It is an elementary exercise in the pigeonhole principle that for any integer $Q \geq 1$ there exist q, $1 \leq q \leq Q^{n-1}$, and integers $p_1, p_2, \ldots p_{n-1}$, such that $|p_j - q\alpha^j| \leq 1/Q$. Thus, for any $\sigma \geq 1$, there is an infinite sequence of positive integers q so that (σ, q, α) is good. On the other hand, none of the resulting approximations are all *that* good: Drmota and Tichy [DT] have shown that for any algebraic number α of degree n, there is a $\sigma > 0$ so that for no q is (σ, q, α) good. Such vectors are termed 'badly approximable'. The scaled error associated with q is

$$q^{1/(n-1)}(p_1 - q\alpha, p_2 - q\alpha^2, \ldots p_{n-1} - q\alpha^{n-1}),$$

and it is bounded between two balls about the origin.

If we were to calculate, for some $\sigma \geq 1$, the first several good q and plot the associated scaled error vectors, we should see some sort of scatterplot of points, all belonging to this hollow ball. Until recently, the calculation of a robust set of good q, for a goodly sample of algebraic α, has been difficult. The LLL algorithm gives passable industrial grade good q. The Jacobi-Perron multidimensional continued fraction algorithm converges rapidly, but the q it gives are not necessarily even *good* q as we have defined them. The literature on the Jacobi Perron algorithm deems the sequence of approximations associated with a sequence of q's to be *strongly convergent* to the approximation target $(\theta_1, \theta_2, \dots \theta_d)$ provided $|(p_1, p_2, \dots p_d - q(\theta_1, \theta_2, \dots \theta_d)| \to 0$. The q we want must give approximations involving an error that is roughly $O(q^{-1/d})$ times that of the acceptable error in strong convergence.

In 1994, Lagarias [La] gave a description and analysis of his geodesic multidimensional continued fraction algorithm. This algorithm takes as input a vector $\theta \in \mathbb{R}^n$, and returns a sequence of unimodular matrices, the rows of which are increasingly near to parallel to the original θ. The first row is always, in a certain technical sense (due to Hermite), an optimal approximation, taking into account an appropriate tradeoff between quality of approximation and the size of the q involved.

(An integer $q > 0$ is Hermite optimal for a target $\theta \in \mathbb{R}^n$ if there exist $0 < u < v$ and $\mathbf{p} \in \mathbb{Z}^n$ so that $|\mathbf{p} - q\theta|^2 + t^2q^2$ is less, for all $u < t < v$, than any other $|\mathbf{p'} - q'\theta|^2 + t^2q'^2$. Here, $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x} = \sum_1^n x_j^2$.)

The Lagarias algorithm is of exponential complexity in the dimension, but it is tolerably fast for small dimension, and with current technology, feasible up to dimension ten or so. The author implemented this algorithm in Mathematica and eventually got around to looking at the scatterplots described above, for a variety of cubic algebraic integers, expecting to see a sort of cloud of points, with perhaps some clues to a density function. A scatterplot of the errors associated with the first several Hermite q for $(\log 2, \log 3)$ is representative of what one might have expected to see, except that in this case, there is no evidence of the 'hole in the middle' that must occur in the case of badly approximable pairs.

Fig. 1.1 Scaled error vectors for a pair of transcendentals.

But in every case, the sequence of 'Hermite optimal' q generated by the algorithm gave rise to a scatterplot in which all the points appeared to lie on one or several concentric, similar ellipses or hyperbolas about the origin; with ellipses when α had a pair of complex conjugates, and hyperbolas when α was totally real. The figures at the end of this section show typical results, for $x^3 - 7x - 11$ and for $x^3 - 7x - 3$, respectively.

Numerical work confirmed that the curves really were (very nearly) ellipses and hyperbolas, and the hunt was on for an explanation. In this work we explain why this was observed, and extend the analysis to the general case of α an algebraic integer of degree $n > 1$. There is a second result concerning the good approximations. For σ sufficiently large, if $q[k, \sigma, \alpha]$ denotes the kth successive $q \geq 1$ so that (σ, q, α) is good, then there is a finite set of numbers so that $q[k+1, \sigma, \alpha]/q[k, \sigma, \alpha]$ lies within $O(q^{-n/(n-1)})$ of some element of that set, as $q \to \infty$. So, even as the scaled errors fall nearly on a finite set of surfaces, the ratios of the underlying q's fall nearly on a finite set of numbers.

When β is an algebraic *number*, but not an algebraic integer, of degree n, then there is a positive integer $N = N[\beta]$ so that $N\beta$ is an algebraic integer. Any particularly good approximation $(p_1/q, p_2/q, \ldots p_{n-1}/q)$ to $(\beta, \beta^2, \ldots \beta^{n-1})$ would yield a comparably good approximation, namely $(Np_1/q, N^2p_2/q, \ldots N^{n-1}p_{n-1}/q)$, to $(N\beta, (N\beta)^2, \ldots, (N\beta)^{n-1})$.

While our analysis and results apply to the case $n = 2$, they give nothing new for that case. The best approximations come from the continued fraction expansion of α. Everything we prove, as well as much that has no evident analogue in higher dimensions, is quite well understood in the case of quadratic irrationals. Still, the analysis here does represent an extension to higher dimensions of many of the properties of the continued fraction expansion of a quadratic irrational.

We illustrate, and foreshadow, with the example $\alpha = \sqrt{3} - 1$, which has continued fraction expansion

$$\alpha = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \ldots}}}}$$

and for which $\mu = 2 + \sqrt{3}$ is a fundamental unit of $\mathbb{Q}(\alpha)$. The minimal polynomial for μ is $x^2 - 4x + 1$, and the associated matrix M for which that polynomial is the minimal polynomial is $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 4 \end{smallmatrix}\right)$. The matrix $P = \left(\begin{smallmatrix} 1 & -1 \\ 2 & 1 \end{smallmatrix}\right)$ has the property that for $k \geq 1$, M^kP has rows nearly parallel to $(1, \alpha)$. Moreover,

(1) The rows of M^kP provide good approximations to the desired (target) direction $(1, \alpha)$.
(2) All 'good' approximations (q, p) to $(q, q\alpha)$ can be written as simple integer linear combinations of the rows of some $(1/3)M^kP$. For in-

stance, $(56, 41)$, associated with $[0, 1, 2, 1, 2, 1, 2, 1]$, can be written as $(-1/3)(26, 19) + (2/3)(97, 71)$, and thus as a combination of the rows of M^3P, while $(7953, 5822)$, another continued fraction convergent of $\sqrt{3} - 1$, is $(1/3)$ times the sum of the rows of M^7P.

(3) The approximation errors $e(q) = q(1, \alpha) - (q, p)$ tend to zero and are always comparable to $1/q$. Thus, to properly assess which q do the best job, it is necessary to rescale the errors by multiplying by q. This done, the scaled errors converge to a number of one-dimensional 'circles' about the origin of a fixed radii. The errors 'rotate' their way around these 'circles', flipping alternately from negative to positive and back, as well as hopping from level to level.

So, what happens in higher dimensions? Calculation of a few hundred consecutive Hermite optimal denominators for some typical cubic algebraic integers, followed by plotting of the scaled errors, gives a clue. Here we show two such plots. These show scaled errors for Hermite optimal denominators associated with (α, α^2) when α is the real root of $x^3 - 7x - 11$, or the largest of the three real roots of $x^3 - 7x - 3$, respectively:

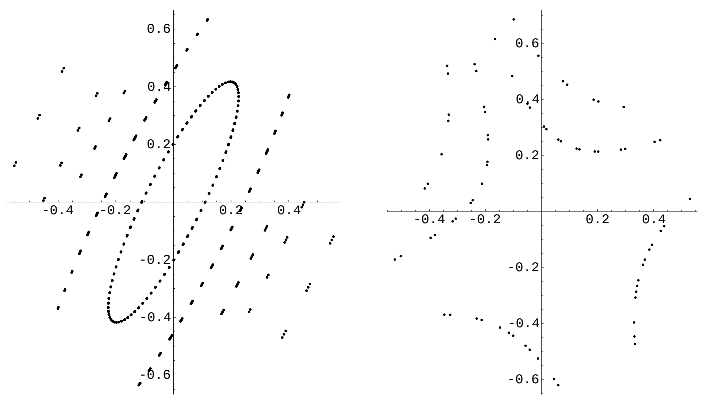

Fig. 1.2 Scaled error vectors for two algebraic integers.

1.2 Outline and Plan of Proof

The full details of the results require additional notation and are thus deferred to the appropriate sections. But we can now indicate the major stepping stones along the path to these results.

(1) Given an algebraic integer α of degree $n > 1$, there is a unit $\mu > 1$ in $\mathbb{Q}(\alpha)$ so that $\mathbb{Q}(\mu) = \mathbb{Q}(\alpha)$. For this unit μ, there is a positive integer $d = d[\mu]$ so that every algebraic integer $\beta \in \mathbb{Q}(\alpha)$ has the form

$$\beta = \frac{1}{d}\sum_{k=0}^{n-1} b_k \mu^k.$$

(This is not to say that all such sums give an algebraic integer. A simple example would be $\mathbb{Q}(\sqrt{5})$, where the algebraic integers have the form $\frac{1}{2}(a + b(\sqrt{5} - 1))$ with a even.)

(2) Let

$$V_\mu := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1^2 & \mu_2^2 & \cdots & \mu_n^2 \\ \vdots & \vdots & \vdots & \vdots \\ \mu_1^{n-1} & \mu_2^{n-1} & \cdots & \mu_n^{n-1} \end{pmatrix}$$

be the Vandermonde matrix associated with $\mu = \mu_1$, where the remaining conjugates of $\mu_2, \ldots, \mu_n$ of μ are numbered so that real conjugates come first, and so that if there are r_1 real conjugates and r_2 complex conjugate pairs, then $\mu_{r_1+r_2+k} = \overline{\mu}_{r_1+k}$ for $1 \le k \le r_2$. Then the $n \times n$ matrix $P_\mu = V_\mu V_\alpha^t$ is nonsingular and has rational integer entries.

(3) There is a sequence $(\nu[k])$, $(k \ge 1)$ of units, each greater than one, in $\mathbb{Q}(\alpha)$, and a positive constant C, so that for all $k \ge 1$, if $\nu[k]_j$ denotes the jth conjugate of $\nu[k]$, numbered consistent with the conjugates of μ, then for $2 \le i, j \le n$,

$$|\nu[k]_i/\nu[k]_j| \le C \text{ and } 2 \le \nu[k+1]/\nu[k] \le C.$$

As the other conjugates of $\nu[k]$ are comparable and have product with absolute value $1/\nu[k]$, each must be $O(\nu[k]^{-1/(n-1)})$.
Since $\sum_{j=1}^n \nu[k]_j = q[k]$ is an integer, and since the other conjugates are small, $q[k] = \nu[k] + O(q[k]^{-1/(n-1)})$.

Remark 1.1 *For k sufficiently large, these units are a special case of PV numbers.*

(4) For any real algebraic integer α of degree $n > 1$, there is a $\sigma_0 > 0$ and a positive integer m so that for every $\sigma > \sigma_0$ there is a finite set

$A(\sigma, \alpha) \subseteq \mathbb{Z}^n$ so that for all q with (σ, q, α) good, there is an $l \geq 1$ and an $\mathbf{a} \in A(\sigma, \alpha) \subset \mathbb{Z}^n$ so that

$$q = \frac{1}{m} \sum_{k=1}^{n} a_k \sum_{j=1}^{n} \mu_j^{k-1} \nu[l]_j.$$

Furthermore, the corresponding $\mathbf{p} \in \mathbb{Z}^{n-1}$ has, for $1 \leq i \leq n-1$,

$$p_i = \frac{1}{m} \sum_{k=1}^{n} a_k \sum_{j=1}^{n} \mu_j^{k-1} \nu[l]_j \alpha_j^{i-1}.$$

This formula will lead to the announced results about the scaled errors associated with good q.

1.3 Proof of the Existence of a Unit $\mu \in \mathbb{Q}(\alpha)$ of Degree n

The set U_α of units in $\mathbb{Q}(\alpha)$ is a finitely generated multiplicative group with structure $\{\pm 1\} \times \mathbb{Z}^r$ with $r = r_1 + r_2 - 1$ where (r_1, r_2) is the signature of $\mathbb{Q}(\alpha)$, that is, where α has r_1 real conjugates and $2r_2$ that are not real. [Co]

Given a fundamental set of units $\{\mu_1, \mu_2, \ldots, \mu_r)\}$ for $\mathbb{Q}(\alpha)$, every unit $\gamma \in \mathbb{Q}(\alpha)$ has the form

$$\gamma = \pm \prod_{j=1}^{r} \mu_j^{a_j}, \qquad (a_1, a_2, \ldots a_r) \in \mathbb{Z}^r.$$

Let $\lambda(\gamma) = (a_1, a_2, \ldots a_r)$ be the list of exponents in this characterization, so that $\lambda : U_\alpha \to \mathbb{Z}^r$. The mapping is two to one, and the inverse image of $\mathbf{0}$ is ± 1. For any subfield K of $\mathbb{Q}(\alpha)$, let $\Lambda(K)$ be the lattice of all $a \in \mathbb{Z}^r$ so that $\lambda^{-1}(a) \subseteq K$. If, contrary to our claim, there were to be a real algebraic integer α so that no one unit μ of $\mathbb{Q}(\alpha)$ generated $\mathbb{Q}(\alpha)$, then for each unit $\mu \in U_\alpha$, the subfield $K(\mu)$ generated by μ would be a proper subfield of $\mathbb{Q}(\alpha)$. From Galois theory, the set $\mathcal{K}$ of proper subfields of $\mathbb{Q}(\alpha)$ is finite. This would lead to a representation of $\mathbb{Z}^r$ as the union of a finite set of lattices $\Lambda(K)$.

In one dimension we are at once done since no finite set of proper sublattices of $\mathbb{Z}$ has union $\mathbb{Z}$; ± 1 would be excluded nor could the density of the union be 1. There are, after all, infinitely many primes. But even in two dimensions, one may readily construct a set of three proper lattices of $\mathbb{Z}^2$, each of determinant 2, so that their union is all of $\mathbb{Z}^2$.

This does not much impede our argument, because, we claim, the sublattices $\Lambda(K) \subset \mathbb{Z}^r$ corresponding to proper subfields K have dimension less than r, and thus density zero in $\mathbb{Z}^r$. From this, the existence of a generating μ is immediate.

To prove the claim, we note that the degree of K over $\mathbb{Q}(\alpha)$ is n/d for some $d > 1$. Thus if K has signature (r_1', r_2') then $r_1' + 2r_2' = n/d \leq n/2$ while $r = r_1 + r_2 - 1 \geq (n-1)/2$. Thus $r_1' + r_2' - 1 < r$, and so the number of generators of U_μ in K is less than r, and the lattice $\Lambda(K)$ has dimension less than r. From this, it is apparent that not only does there exist a unit $\mu \in \mathbb{Q}(\alpha)$ so that $\mathbb{Q}(\mu) = \mathbb{Q}(\alpha)$, but that almost all units are like that. We choose for our fixed μ associated with $\mathbb{Q}(\alpha)$ any such unit that is greater than 1.

In the outline, we mentioned that the matrix $P_\mu = V_\mu V_\alpha^t$ had rational integer entries and was nonsingular. The proof of this claim is simple now that we know that μ has n distinct conjugates: We now know that we can indeed construct V_μ, and the entry (j,k) of P_μ is $P_\mu[j,k] = \sum_{i=1}^n \mu_i^{j-1}\alpha_i^{k-1}$ and is invariant under automorphisms of $\mathbb{Q}(\alpha)$. It is thus both an algebraic integer, and a rational number. Since the conjugates of μ are distinct, V_μ is nonsingular and of course for the same reason so is V_α^t so P_μ is nonsingular.

1.4 The Sequence $\nu[k]$ of Units with Comparable Conjugates

For a positive unit $\nu \in \mathbb{Q}(\alpha)$, we call (C, ν) *good* if, for all conjugates of ν other than ν itself, $|\nu_j| \leq C\nu^{-1/(n-1)}$. If the signature of α is (r_1, r_2), we put $r = r_1 + r_2 - 1$. This is the number of units, apart from -1, needed as generators for the multiplicative group of units in $\mathbb{Q}(\alpha)$. We prove that there exists $C = C(\alpha) > 1$ and a sequence $(\nu[k]), (k \geq 1)$ of units of $\mathbb{Q}(\alpha)$, each of degree n, so that $2 \leq \nu[k+1]/\nu[k] \leq C$ and $(C, \nu[k])$ is good.

We associate to any positive unit $\nu \in \mathbb{Q}(\alpha)$ the real vector

$$\mathbf{x}(\nu) = (x_1, x_2 \dots x_{r+1}) \in \mathbb{R}^{r+1}$$

given by

$$\mathbf{x}(\nu) = (\log(\nu_1), \log|\nu_2|, \dots \log|\nu_{r_1}|, 2\log|\nu_{1+r_1}|, \dots 2\log|\nu_{r_1+r_2}|),$$

which lies in the hyperplane $H = \{x \in \mathbb{R}^{r+1} : \sum_1^{r+1} x_k = 0\}$.

Now consider the lattice $\Lambda(\alpha) \subset H$ consisting of all $\mathbf{x}(\nu)$ so that ν is a unit of $\mathbb{Q}(\alpha)$. We claim that there is a set of r such units, each of degree

n, which generates the group of units of $\mathbb{Q}(\alpha)$ apart from the torsional component ± 1. Only the claim that we may take our units to have degree n requires proof. The lattice points $\mathbf{a}$ in the lattice of the previous section, corresponding to units of degree less than n, have zero density in $\mathbb{Z}^r$ and lie in a finite set of sub-dimensional hyperplanes. Thus if we select, as we may, a lattice basis of $\mathbb{Z}^r$ consisting of vectors all nearly parallel to some vector not in any of the hyperplanes, the units corresponding to that lattice basis will all have degree n. This proves the claim.

Thus in the representation of units as $\nu = \pm \prod_1^r \mu[j]^{a_j}$, we may as well, and we do, assume that all the $\mu[j]$ have degree n, and each is greater than 1. Each of these units has a list of n conjugates, $\mu[j] = \mu[j]_1, \mu[j]_2, \ldots \mu[j]_n$, which we number consistently with the numbering of the conjugates of α. We take our basic μ, used in V_μ and so on, to be $\mu[1]$, and we define $\delta[j] \in \mathbb{R}^{r+1}$ by

$$\delta[j] = (\log|\mu[j]_1|, \ldots, \log|\mu[j]_{r_1}|, 2\log|\mu[j]_{1+r_1}|, \ldots 2\log|\mu[j]_{r_1+r_2}|).$$

The set $\Delta = \{(\delta[1], \delta[2], \ldots, \delta[r]\}$ is then a basis for H. The question of the existence of units ν fitting our conditions, can now be rephrased as a question about the lattice $L \subset H$ generated by Δ: Does there exist a sequence $\phi[k](k \geq 1)$ of elements in L, confined to a cylinder about the ray $\{t(r,-1,-1,\ldots-1) : t > 0\}$of radius small enough that any ν corresponding to a point inside the cylinder must have (C,ν) good, and so that the sequence of first components of $\phi[k]$ is increasing to infinity with $\log 2 \leq \phi[k+1]_1 - \phi[k]_1 \leq \log C$?

Put this way, a simple construction shows that the answer is yes when C is large enough. Let $\Delta = \Delta(L) := \{\sum_1^n s_j\delta[j] : -1/2 < s_j \leq 1/2 \text{ for } 1 \leq j \leq n\}$. Let $\theta > 1$ be large enough that the first coordinate of any element of $\theta(r,-1,\ldots,-1) + \Delta$ is greater than the first coordinate of any element of Δ. Any translate of Δ contains exactly one element of L. We take $\mathbf{x} \in L$ to be an element sufficiently distant from the cylinder axis that no element of $\Delta + \mathbf{x}$ lies on the cylinder. (This displacement of the cell from the axis forecloses the possibility that the sequence of lattice elements would lie in some fixed subspace of dimension less than that of H.) We then take $\nu[k]$ to be the unit corresponding to the element $\phi[k]$ of L in $\Delta + x + k\theta(r,-1,\ldots,-1)+$. The units then satisfy $\nu[k] = \exp(kr\theta + O(1))$, $2 \leq \nu[k+1]/\nu[k] \leq C$, and $|\nu[k]_j/\nu[k]_i| < C$ for $2 \leq i,j \leq n$, as claimed. Furthermore, the set of ratios $\nu[k+1]/\nu[k]$ is finite because the set of differences $\phi[k+1] - \phi[k]$ is a subset of the set of all lattice points in

$\theta(r, -1, \ldots -1) + 2\Delta$ and there are just 2^n such lattice points. Finally, we purge the sequence of any elements of degree less than n; as there are only finitely many subfields of $\mathbb{Q}(\alpha)$ to avoid, and as each corresponding subspace of H can touch only finitely many of the boxes from which we originally took our sequence of lattice points, this leaves an infinite sequence of good units, each of degree n, and with all the asymptotic properties of the original sequence.

1.5 Good Units and Good Denominators

Let $\nu = \nu_k$ be one of our sequence of good units. For each such ν, there is a polynomial f_ν with integer coefficients and degree less than n, so that $\nu = \frac{1}{d} f_\nu(\mu)$.

From this point on, we dispense with keeping track of the details of the constants in good units or good denominators, and use 'Big Oh' terminology. When we say that some object depending on k is good, what we mean is that there exist constants so that for all sufficiently large k, that object is good with respect to those constants, and 'O' will also be with respect to k.

Now for any $\beta \in \mathbb{Q}(\alpha)$, let D_β be the diagonal matrix with diagonal entries $D_\beta[i,i] = \beta_i$, the ith conjugate of β. Let

$$M_\nu = V_\mu D_\nu V_\mu^{-1} P_\mu = V_\mu D_\nu V_\alpha^t.$$

Now the (k,j) entry of M_ν may be written as $M_\nu[j,k] = \sum_{i=1}^n \mu_i^{j-1} \nu_i \alpha_i^{k-1}$ which is a sum of products of integers of $\mathbb{Q}(\alpha)$ and thus an integer of $\mathbb{Q}(\alpha)$, and since the sum is a symmetric polynomial in $\alpha_1, \ldots, \alpha_n$, it is a rational number. Thus, the entries of M_ν are integers. Since both V_μ and V_ν have determinants of absolute value 1, $|\det M_\nu| = |\det P_\mu| = p$ say, and $p \geq 1$.

On the other hand, the rows $M\nu[j]$ of M_ν satisfy the estimate

$$M_\nu[j] = \mu^{j-1} \nu (1, \alpha, \alpha^2, \ldots, \alpha^{n-1}) + O(\nu^{-1/(n-1)}).$$

In particular, the $(1,1)$ entry q_ν, which is also (for large ν) the integer nearest the PV number ν, is a good denominator for simultaneous diophantine approximation of the powers of α.

Now suppose $\mathbf{c} = (c_1, c_2, \ldots c_n) \neq 0 \in \mathbb{Z}^n$. Then $\sum_1^n c_k \mu^{k-1} \neq 0$. Thus for any such $\mathbf{c}$ and for any $\nu = \nu_k$ from our sequence of good units,

$$\mathbf{c} \cdot M_\nu = \nu \sum_{k=1}^{n} c_k \mu^{k-1}(1, \alpha, \alpha^2, \ldots, \alpha^{n-1}) + O(\nu^{-1/(n-1)})$$

so that the first entry of this vector is a good denominator for α. More important is that a kind of converse is also true.

Theorem 1.1 *Let α be a real algebraic number of degree $n > 1$. Let $\mu > 1$ be a unit of $\mathbb{Q}(\alpha)$ of degree n. Let $p = |\det P_\mu|$. Suppose (ν) is a sequence of units of degree n in $\mathbb{Q}(\alpha)$, with the property that if $\nu_j[k]$ are the conjugates of ν_j, ordered so that $\nu_j = \nu_j[1]$, then $\nu_j[1] > 1$, $2 < \nu_{k+1}/\nu_k < C$ for all k, and $|\nu_j[k]| \in [1/C, C]\nu_j^{-1/(n-1)}$ for all conjugates $\nu_j[k]$ of ν_j, $2 \le j \le n$. Let q_k be the sum of ν_k and all its conjugates. Then for all σ large enough that (σ, q_k, α) is good for all k, there exists $N \ge 1$ so that if $q \ge 1$ and (σ, q, α) is good, there exist integers $(c_1, c_2, \ldots, c_n)$ with $|c_i| \le N$ for $1 \le i \le n$, not all zero, and an integer $m \ge 1$, so that $\nu_m < q < \nu_{m+1}$ and, writing $\nu_m = \nu$,*

$$q = \frac{1}{p}\left(\mathbf{c} \cdot M_\nu\right)_1 .$$

Remark 1.2 *Informally, this says that all good denominators for α come from simple rational linear combinations of the rows of some M_ν, for some good unit ν in our sequence. Stripped to a mnemonic, good denominators come from good units.*

Proof. Suppose (σ, q, α) is good. Then the integers nearest $q\alpha_j$ respectively, call them $(p_1, \ldots, p_{n-1})$, satisfy $|p_j - q\alpha^j| \le \sigma q^{-1/(n-1)}$ for $1 \le j \le n-1$. Choose m so that $\nu = \nu_m < q < \nu_{m+1}$. Note that M_ν has integer entries, but on the other hand, we can express the kth row of M_ν in the form

$$M_\nu[k] = (1 + O(q^{-1/(n-1)}))q_m \mu^{k-1}(1, \alpha, \alpha^2, \ldots \alpha^{n-1}) + \mathbf{e}$$

where $\mathbf{e} \cdot (1, \alpha, \alpha^2, \ldots \alpha^{n-1}) = 0$ and where $|\mathbf{e}| = O(q^{-1/(n-1)})$.

Now consider the $n \times n$ matrix T_q determined by the twin requirements that $(1, \alpha, \alpha^2, \ldots \alpha^{n-1}) \cdot T_q = q^{-1}(1, \alpha, \alpha^2, \ldots \alpha^{n-1})$ and that for any vector $\mathbf{x} \in \mathbb{R}^n$ with $\mathbf{x} \cdot (1, \alpha, \alpha^2, \ldots \alpha^{n-1}) = 0$, $\mathbf{x} \cdot T_q = q^{1/(n-1)}\mathbf{x}$. From the foregoing observations about M_ν, it follows that the rows of $M_\nu \cdot T_q$ have length $O(1)$. Since $\det(T_q) = 1$, it follows that the rows of $M_\nu \cdot T_q$ are in fact comparable to 1 in length, and from this, it follows that $(M_\nu \cdot T_q)^{-1}$ has bounded entries, independent of m and q. Now let

$\mathbf{r} = \mathbf{r}[\nu] = (r_1, r_2, \ldots, r_n) = (q, p_1, p_2, \ldots, p_{n-1}) \cdot M_\nu^{-1}$. Then pr_j is an integer for all j. Also, $(q, p_1, p_2, \ldots, p_{n-1}) \cdot T_q = O(1)$ and $\|(M_\nu \cdot T_q)^{-1}\| \ll 1$, so $|(q, p_1, p_2, \ldots, p_{n-1}) \cdot T_q \cdot T_q^{-1} M_\nu^{-1}| \ll 1$. That is, $|r_j|$ is bounded independent of m and q. From this it follows that $(q, p_1, p_2, \ldots, p_{n-1}) = p^{-1}\mathbf{c} \cdot M_\nu$ with $c_j = Pr_j \in \mathbb{Z}$, and the $|c_j|$ are bounded, independent of m and q. The bound does depend on α, μ, and σ. This completes the proof of the theorem. In a nutshell, every good q is the first entry in an integer vector of the form $p^{-1}\mathbf{c} \cdot M_\nu$ for some $\nu = \nu[m]$ in our sequence of good units, and $\mathbf{c}$ with $\mathbf{c} \in \mathbb{Z}^n$ bounded, independent of m. □

1.6 Ratios of Consecutive Good q

We are now in position to prove our second theorem, to the effect that the ratios of consecutive good q are asymptotically close to one or another of a finite set of elements of $\mathbb{Q}(\alpha)$.

Theorem 1.2 *Let α be a real algebraic integer of degree $n > 1$. Then for all sufficiently large σ, there exists a finite set $B(\sigma, \alpha) \subset \mathbb{Q}(\alpha) \cap (1, \infty)$, and a constant $C = C(\sigma, \alpha) > 1$, so that if (σ, q, α) and (σ, q', α) are consecutive good triples for α, then there exists $\beta \in B(\sigma, \alpha)$ so that*

$$|\beta - q'/q| \leq Cq^{-n/(n-1)}.$$

Proof. Choose σ so that for all q_k associated with ν_k, (σ, q_k, α) is good. Choose N so that, for all q such that (σ, q, α) is good, there exists a $\mathbf{c} \in [-N, N]^n$ and a $k \geq 1$ such that q is the first entry of $p^{-1}\mathbf{c} \cdot M_{\nu_k}$. Choose m so that $q_m \leq q < q_{m+1}$. Then $q' \leq q_{m+1}$. Let $\nu = \nu_m$. Then as in Theorem 7.1, there also exists $\mathbf{c}' \in [-N, N]$ such that

$$q' = \frac{1}{p} \left(\mathbf{c}' \cdot M_\nu\right)[1] \; .$$

Furthermore,

$$(\mathbf{c} \cdot M_\nu)[1] = \nu \sum_{j=1}^{n} \mathbf{c}_j \mu^{j-1} + O(\nu^{-1/(n-1)}),$$

and similarly for $\mathbf{c}'$. Thus, since $q_m = \nu + O(\nu^{-1/(n-1)})$,

$$(\mathbf{c} \cdot M_\nu)[1] = q_m \sum_{j=1}^{n} c_j \mu^{j-1} + O(q^{-1/(n-1)}),$$

and similarly for $\mathbf{c}'$ and q'. Thus

$$\frac{q'}{q} = \frac{\sum_{j=1}^{n} \mathbf{c}'_j \mu^{j-1} + O(q^{-1/(n-1)})}{\sum_{j=1}^{n} \mathbf{c}_j \mu^{j-1} + O(q^{-1/(n-1)})}.$$

Now the entries of $\mathbf{c}$ and $\mathbf{c}'$ are bounded by N, so the quantity $|\sum_{j=1}^{n} \mathbf{c}_j \mu^{j-1}|$ is bounded and bounded away from zero, independently of m, as is the corresponding quantity for $\mathbf{c}'$. Thus

$$\frac{q'}{q} = \frac{\sum_{j=1}^{n} c'_j \mu^{j-1}}{\sum_{j=1}^{n} c_j \mu^{j-1}} + O(q^{-n/(n-1)}).$$

The set $B(\sigma, \alpha)$ of ratios of this type is finite because $\mathbf{c}$ and $\mathbf{c}'$ come from a finite set. □

1.7 The Surfaces Associated With the Scaled Errors

In this section we explain the experimentally observed ellipses and hyperbolas mentioned in the introduction, and prove a theorem generalizing what was observed.

Any hyperbola centered at the origin has a parametric representation as the image, under a nonsingular linear transformation of $\mathbb{R}^2$, of $\{(t_1, t_2) : t_1 t_2 = 1\}$, and any ellipse is a like transformation of the unit circle. For $(r_1, r_2) \in \mathbb{Z}^2$ with $r_1 + 2r_2 = n$ and $r_1 > 0, r_2 \geq 0$, let

$$U[r_1, r_2] = \{(z_2, z_3, \dots z_n) \in \mathbb{C}^{n-1} : z_j \in \mathbb{R}, 2 \leq j \leq r_1,$$
$$z_{r_1+r_2+k} = \overline{z}_{r_1+k}, 1 \leq k \leq r_2, \text{ and } \prod_{j=2}^{n} |z_j| = 1\}.$$

Also, if M is an $(n-1) \times (n-1)$ invertible matrix with complex entries, and if in each column of M, the first $r_1 - 1$ entries are real, and the last r_2 entries are the respective conjugates of the middle r_2 entries, we say that M is of *type* (r_1, r_2).

If M has type (r_1, r_2), then $U[r_1, r_2]M$ is a surface in $\mathbb{R}^{n-1}$ of dimension $n-2$, and $0 \notin U[r_1, r_2]M$. In the special case $r_1 = 3$, $r_2 = 0$, it is a pair of hyperbolas, a linear transformation of $\{(t_1, t_2) : t_1 t_2 = \pm 1\}$, whereas if $r_1 = r_2 = 1$, it is an ellipse.

Theorem 1.3 *Let α be an algebraic integer of degree $n > 1$ and signature (r_1, r_2). Then for all $\sigma > 1$ there is a finite set $R(\sigma, \alpha)$ of positive elements* *of $\mathbb{Q}(\alpha)$, and a nonsingular matrix W_α of type (r_1, r_2), so that for each $q \geq 1$ with (σ, q, α) good, there is a $\gamma \in R(\sigma, \alpha)$ such that the scaled error*

$$q^{1/(n-1)}\left[(p_1, \ldots p_{n-1}) - q(\alpha, \ldots \alpha^{n-1})\right]$$

lies, to within $O(q^{-1/(n-1)})$, on the surface $\gamma U[r_1, r_2]W_\alpha$. The (k, l) entry of W_α is $W_\alpha[k, l] = \alpha_{k+1}^l - \alpha_1^l$.

Proof. Without loss of generality we may take σ arbitrarily large, because if (σ_1, q, α) is good and $\sigma_2 > \sigma_1$, then (σ_2, q, α) is good. We may also take q arbitrarily large. We recall that $p = |\det P_\mu|$. Now by Theorem 7.2, there is a finite set $B(\sigma, \alpha)$, and a positive integer N, so that for all large q with (σ, q, α) good,

$$(q, p_1, \ldots p_{n-1}) = \frac{1}{p}(\mathbf{c} \cdot M_\nu)$$

for some $\nu = \nu[m]$ with $m \geq 1$ and some integer-entried $\mathbf{c} \in [-N, N]^n$. Now $M_\nu = V_\mu D_\nu V_\alpha^t$. A routine calculation then yields

$$(\mathbf{c} \cdot M_\nu)_j = \sum_{i=1}^{n} c_i \sum_{k=1}^{n} \mu_k^{i-1} \nu_k \alpha_k^{j-1}.$$

Thus

$$q = \frac{1}{p} \sum_{i=1}^{n} \sum_{k=1}^{n} c_i \mu_k^{i-1} \nu_k$$

so that the jth entry of $q(\alpha, \alpha^2, \ldots \alpha^{n-1})$ is

$$q\alpha^j = \frac{1}{p} \sum_{i=1}^{n} \sum_{k=1}^{n} c_i \mu_k^{i-1} \nu_k \alpha^j, \ (1 \leq j \leq n-1).$$

The resulting approximation error

$$\mathbf{e}(q) = (p_1, \ldots, p_{n-1}) - q(\alpha, \ldots, \alpha^{n-1})$$

has jth entry

$$e_j(q) = \frac{1}{p}\left(\sum_{i=1}^{n} \sum_{k=2}^{n} c_i \mu_k^{i-1} \nu_k (\alpha_k^j - \alpha_1^j)\right), \ (1 \leq j \leq n-1).$$

This sum can be written as a sum of matrix products, writing $D(x_1, x_2, \ldots)$ for the diagonal matrix based on the given entries:

$$\mathbf{e}(q) = \frac{1}{p} \sum_{i=1}^{n} c_i(\nu_2, \ldots, \nu_n) \cdot D(\mu_2, \ldots, \mu_n)^{i-1} W_\alpha,$$

where $W_\alpha[k, j] = (\alpha_{k+1}^j - \alpha_1^j)$ for $1 \le j, k \le n-1$. We now show that W_α is nonsingular. For if $\mathbf{x} \cdot W_\alpha = 0$, consider $\mathbf{y} = (-\sum_1^{n-1} x_j, x_1, x_2, \ldots, x_{n-1}) \in \mathbb{R}^n$, and consider the $n \times n$ matrix $V_\alpha - V'$, where V' has n identical rows, each being $(1, \alpha, \ldots, \alpha^{n-1})$. Then $\mathbf{y} \cdot (V_\alpha - V') = 0$. But $\mathbf{y} \cdot V' = 0$, so $\mathbf{y} \cdot V_\alpha = 0$ so that $\mathbf{y} = \mathbf{0}$ and thus $\mathbf{x} = \mathbf{0}$.

For $\mathbf{c} \in \mathbb{Z}^n$, let $\xi(\mathbf{c}) = \left(\sum_1^n c_i \mu_1^{i-1}\right)^{1/(n-1)}$, and let

$$S_\alpha(\mathbf{c}) = \{\mathbf{c} \cdot V'_\mu D(s_2, s_3, \ldots s_n) : s \in U[r_1, r_2]\}$$

where V'_μ is the $n \times (n-1)$ matrix got by deleting the leftmost column from V_μ. Then

$$q^{1/(n-1)} \mathbf{e}(q) = (1 + O(q^{-n/(n-1)})) p^{-1/(n-1)} \nu^{1/(n-1)} \xi(\mathbf{c}) \mathbf{e}(q).$$

Now

$$\nu^{1/(n-1)} \mathbf{e}(q) W_\alpha^{-1} = p^{-1} \sum_{i=1}^{n} c_i(\nu_2 \nu^{1/(n-1)}, \ldots, \nu_n \nu^{1/(n-1)}) D(\mu_2, \ldots, \mu_n)^{i-1},$$

and

$$(\nu_2 \nu^{1/(n-1)}, \ldots, \nu_n \nu^{1/(n-1)}) \in U[r_1, r_2].$$

Equivalently, $\nu^{1/(n-1)} \xi(\mathbf{c}) \mathbf{e}(q) W_\alpha^{-1} \in \xi(\mathbf{c}) S_\alpha(\mathbf{c})$, and so the scaled errors lie, to within a factor of $(1 + O(q^{-n/(n-1)}))$, on $p^{-1/(n-1)} \xi(\mathbf{c}) S_\alpha(\mathbf{c}) W_\alpha$.

It remains only to show that for each nonzero $\mathbf{c} \in \mathbb{Z}^n$ there is a nonzero scaling coefficient $\rho(\mathbf{c})$ so that every point of $\xi(\mathbf{c}) S_\alpha(\mathbf{c})$ belongs to $\rho(\mathbf{c}) S_\alpha(1, 0, \ldots, 0)$. The scaling coefficient must be

$$\rho(\mathbf{c}) = \left(\prod_{j=1}^{n} |f_{\mathbf{c}}(\mu_j)| \right)^{1/(n-1)} = \left| \mathcal{N} \left(\sum_{1}^{n} c_j \mu_1^{j-1} \right) \right|^{1/(n-1)}$$

where $f_{\mathbf{c}}(x) = \sum_{k=1}^n c_k x^{k-1}$. Note that $\rho(\mathbf{c}) \ne 0$, because $f_{\mathbf{c}}(\mu) \ne 0$ since μ is algebraic of degree n, and because the other conjugates of μ are likewise of degree n.

To prove that these surfaces are scalar multiples of each other, it is sufficient to show that for any $s \in U[r_1, r_2]$ and any $\mathbf{c} \neq \mathbf{0}$, there exists $t \in U[r_1, r_2]$ so that

$$\mathbf{c} \cdot V'_\mu D(s_2, \dots s_n) = (1, 0, , \dots, 0) \cdot V'_\mu D(t_2, \dots, t_n).$$

So suppose $\mathbf{c} \in \mathbb{Z}^n$, $\mathbf{c} \neq (0, 0, \dots, 0)$ and suppose $s \in U[r_1, r_2]$. Then $\rho(\mathbf{c}) \neq 0$, and

$$\mathbf{c} \cdot V'_\mu D(s) = (f_\mathbf{c}(\mu_2), \dots, f_\mathbf{c}(\mu_n)) D(s) = \rho(\mathbf{c})(s'_2 f_\mathbf{c}(\mu_2), \dots, s'_n f_\mathbf{c}(\mu_n))$$

where $s'_j = s_j / \rho(\mathbf{c})$. But

$$(s'_2 f_\mathbf{c}(\mu_2), s'_3 f_\mathbf{c}(\mu_3), \dots, s'_n f_\mathbf{c}(\mu_n)) \in U[r_1, r_2]$$

so $\mathbf{c} V'_\mu D(U[r_1, r_2]) \subseteq \rho(\mathbf{c}) U[r_1, r_2]$. □

A direct illustration of Theorem 7.3 in three or more dimensions is far more difficult than in two dimensions. The two dimensional paper does not readily reveal nested three dimensional surfaces. But what can be done without great trouble is to calculate the ρ associated with each q, that is, by what number must we multiply the standard surface so that the current scaled error falls on the surface? This was done for $x^4 - 4x + 2$, and the resulting plot is shown in the next figure.

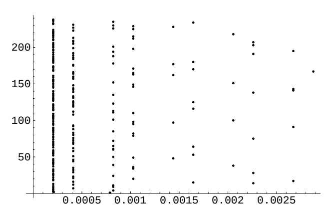

Fig. 1.3 Point frequencies for the various nested surfaces.

1.8 The General Case of Algebraic Numbers in $\mathbb{Q}(\alpha)$

Suppose now that α is as before an algebraic integer of degree n, and a real number, and it has signature (r_1, r_2). Suppose also that

$$\{1, \beta_1, \beta_2, \dots, \beta_{n-1}\} \subset \mathbb{Q}(\alpha)$$

is linearly independent over $\mathbb{Q}$. Recall that (q, σ, β) is *good* if there exist integers $p_1, \ldots, p_{n-1}$ so that $|q\beta_j - p_j| \leq q^{-1/(n-1)}\sigma$ for $1 \leq j \leq n-1$. Then we have the same theorems for $\beta = (\beta_1, \beta_2, \ldots, \beta_{n-1})$ that we had for algebraic integers α.

We first note that there is an invertible matrix B with integer entries, and a positive integer N, such that

$$(1, \beta_1, \ldots, \beta_n) = N^{-1}(1, \alpha, \ldots, \alpha^{n-1})B.$$

Let $d = |\det B|$, so that $B^{-1} = d^{-1}B'$ with B' again an invertible matrix with integer entries. If (q, σ, β) is good, then $(Nq, N^{n/(n-1)}\sigma, \beta)$ is also good, so

$$(Nq, Nq\beta) = (Nq, Np_1, \ldots, Np_{n-1}) + (0, e_1, \ldots, e_{n-1})$$

where $|e_j| \leq N^{n/(n-1)}\sigma(Nq)^{-1/(n-1)}$ for $1 \leq j \leq n-1$. Thus with $b' = \max_{i,j} |B'_{i,j}|$, we have

$$(Nq, Nq\beta)B' = r + (0, e'_1, e'_2, \ldots, e'_{n-1})$$

where $r \in \mathbb{Z}^n$ and where $|e'_j| \leq (nN^{n/(n-1)}\sigma b')(Nq)^{-1/(n-1)}$. Now the first entry of r is dq, so we restate this as

$$|e'_j| \leq (nNd^{1/(n-1)}\sigma b')(dq)^{-1/(n-1)}.$$

Thus, $(dq, (nNd^{1/(n-1)}\sigma b'), \alpha)$ is good. Thus, Theorem 7.3 applies and $(dq)^{1/(n-1)}(e'_1, e'_2, \ldots, e'_{n-1})$ lies, to within $O(q^{-1/(n-1)})$, on a surface of the form $\rho U[r_1, r_2]W_\alpha$, where ρ is one of a finite set $R[\sigma', \alpha]$ of positive real numbers. Let B^* be what remains of B after deleting the top row and the left column. Then $|\det B^*| = d/N$, and B^* is an invertible matrix with integer entries. Now $(e_1, \ldots, e_{n-1}) = d^{-1}e'B^*$, so $e \in q^{-1/(n-1)}d^{-n/(n-1)}\rho U[r_1, r_2]W_\alpha B^* + O(q^{-2/(n-1)})$, from which we conclude that the scaled approximation errors for $(q\beta)$ lie nearly on one of a finite list of surfaces of the same sort as before.

Our theorem about ratios of consecutive good approximations also extends to this nominally more general case. There is a finite set $G(\sigma, \beta) \subset \mathbb{Q}(\alpha) \cap (1, \infty)$, and a constant $C = C(\sigma, \beta) > 1$, such that if q and q' are consecutive good numbers in the context of $(*, \sigma, \beta)$, then there exists $\gamma \in G(\sigma, \beta)$ such that $|q'/q - \gamma| \leq Cq^{-n/(n-1)}$.

To see this, we first note that as before, $(Nq, N^{n/(n-1)}\sigma, \beta)$ as well as $(Nq', N^{n/(n-1)}\sigma, \beta)$are good, as are (dq, σ', α) and (dq', σ', α), where $\sigma' = nNd^{1/(n-1)}b'$

Now choose m_1 and m_2 so that $\nu[m_1] < q < \nu[1+m_1]$ and $\nu[m_2] < q' < \nu[1+m_2]$. Because β is badly approximable, there is a number $K > 1$ such that for all q, q' that are consecutive good (in context of σ) denominators, $q'/q < K$. Now

$$\frac{q'}{q} = \frac{q'}{q(\nu[m_2])} \cdot \frac{q(\nu[m_2])}{q(\nu[m_1])} \cdot \frac{q(\nu[m_1])}{q}.$$

But the the two ratios at either end of this product of ratios are covered by our earlier theorem on ratios of q's, while the central ratio is a ratio that, to our usual tolerances, coincides with $\nu[m_2]/\nu[m_1]$. But this ratio is an algebraic integer corresponding to a bounded lattice displacement, from one lattice point in our cylinder to another. The number of choices for it is therefore finite.

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