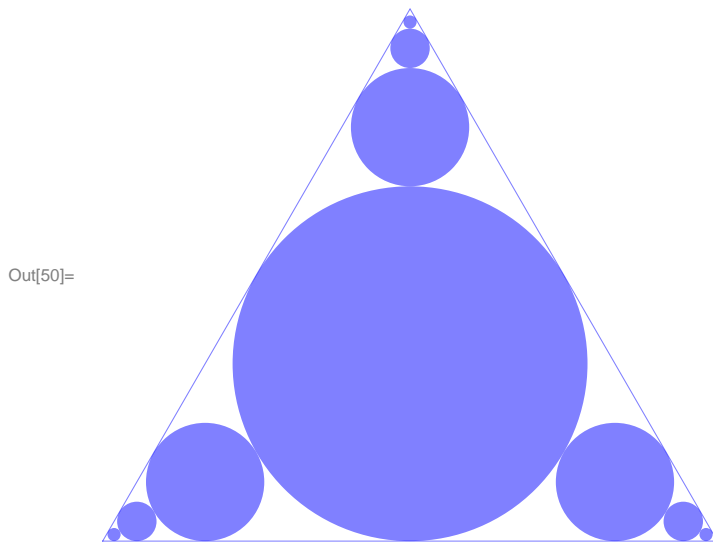


The geometry of an algebraic take on geometric series.

How to add up infinitely many areas and not run out of ink, paper, or patience.

There's a problem in the engineering calculus textbook we use at TAMU, and it reads as follows:

In the figure there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching the other circles and sides of the triangle. If the triangle has sides of length 1, find the total area occupied by the circles.



From the context of the problem, one suspects that the intended solution involved the summation of a geometric series. This would certainly do the trick, and geometric series was one of the topics in the chapter in which this problem appeared.

Well, why do they call them geometric series? There's some geometry to it, at least from a certain point of view. This goes back all the way to Archimedes,

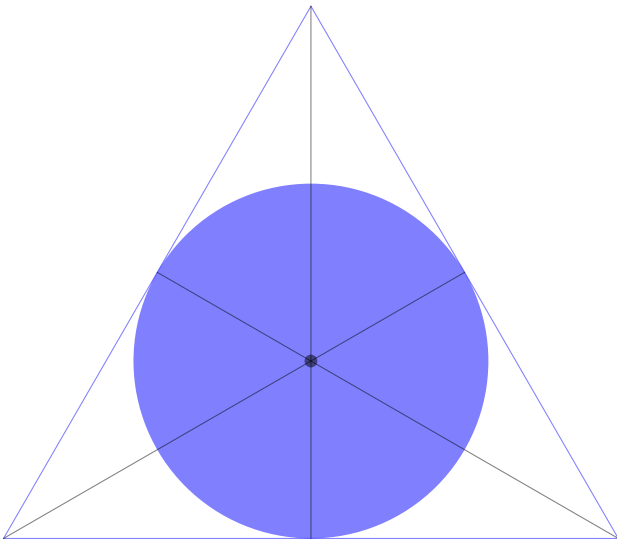
http://en.wikipedia.org/wiki/Geometric_series#Archimedes.27_quadrature_of_the_parabola

and a statement equivalent to the modern understanding of geometric series appears in Euclids "Elements".

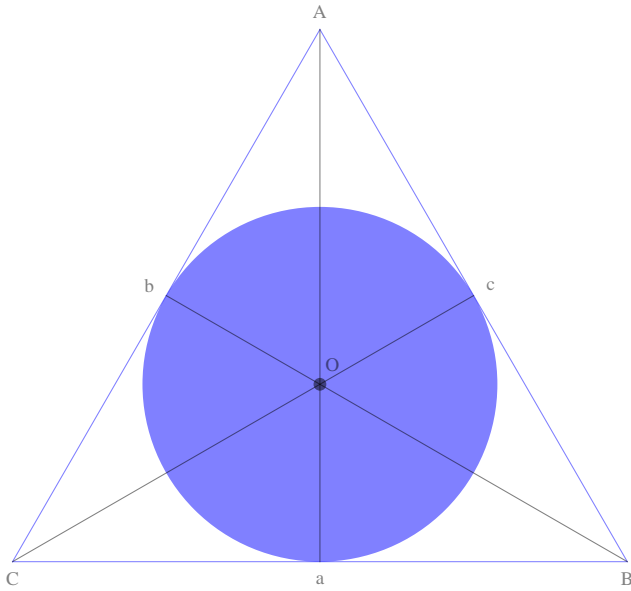
Just for the record, here's one version of the formula: if $r < 1$, then a series starting with a first term A and with each subsequent term being r times the preceding term sums to $A/(1-r)$. In algebraic form, $S = \sum_{k=0}^{\infty} A r^k = \frac{A}{1-r}$.

If we leave aside questions of rigor, as we certainly may in high school, or for that matter, occasionally also in engineering calculus II, the proof is simple enough. S is the sum of its first term together with the rest of them. The first term is A . The rest of them are each smaller, by a factor of r , than the correspondingly placed term in the original S . Thus the sum of the rest of S is equal to rS . That is, $S = A + rS$. So $S - rS = A$, and $S(1-r) = A$ and thus $S = A/(1-r)$.

But the devil is in the details. How does THIS problem have anything to do with geometric series? One of the basic habits one must cultivate in order to get anywhere with geometry is to draw new things into the picture. Here, lines have been introduced from each of the corners to the middle of the opposite side. It is well known that these lines intersect at the centroid of the triangle [this holds whether or not it's equilateral] and that for each line, two thirds of its length lies in the segment between the vertex and the centroid, while the other third of its length continues on from the centroid to the midpoint of the opposite side.



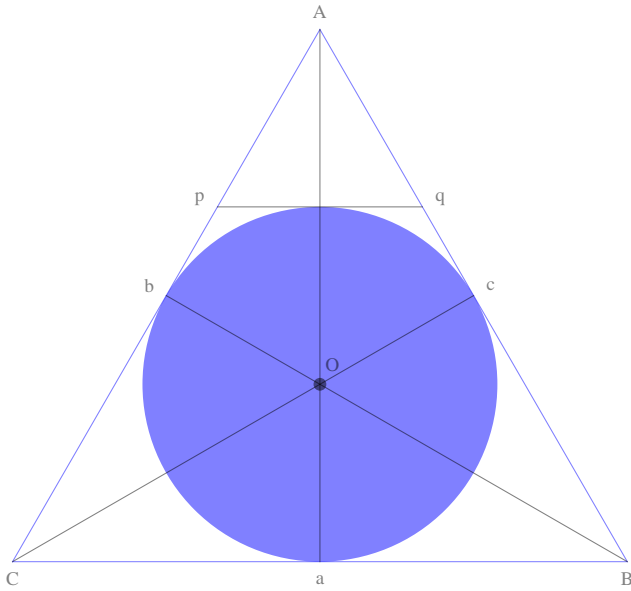
Another good habit is to introduce labels for the things in the picture, and while we're at it, though this is not strictly speaking geometry, it seems to pay to mix geometry with algebra and switch back and forth between the two.



Now we can talk! In triangle CaA , there is a right angle at a , and $|Ca|=1/2$, while $|CA|=1$, so $|Aa|=\sqrt{3}/2$. Now $|Oa|$ is $1/3$ of that, so the radius of our circle is $|Oa|=\sqrt{3}/6$. The area of the shaded circle is thus $\pi/12$.

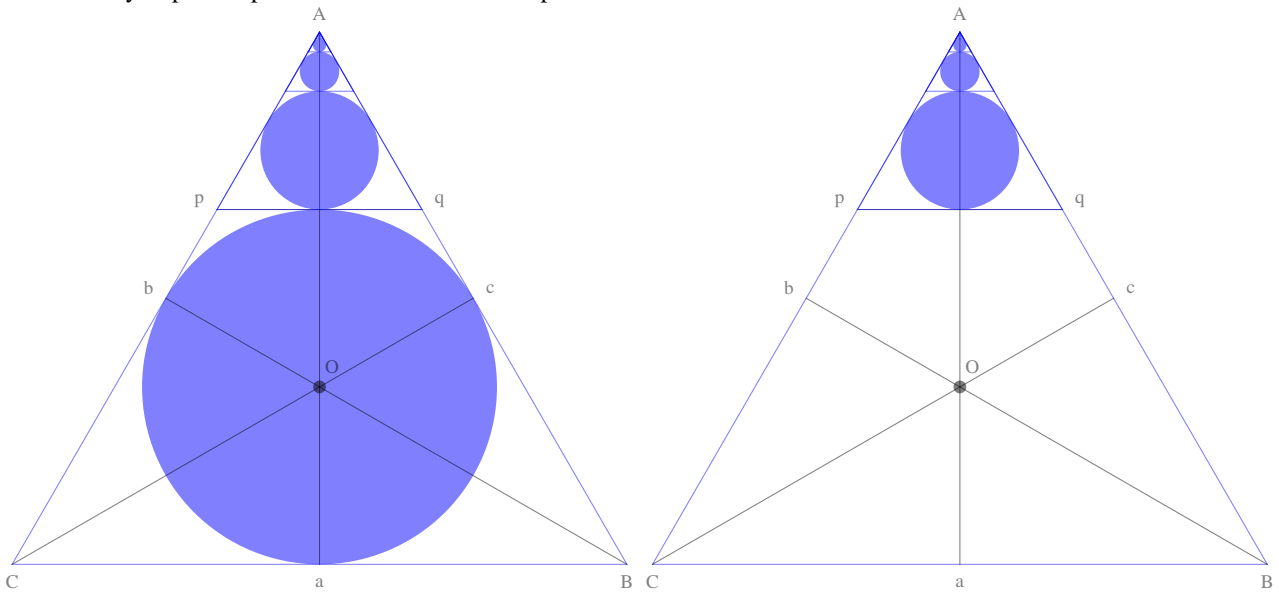
But what of all the other circles? Well, there are three batches of them, all alike. If we can get the area of one batch, we'll basically be done.

Let's introduce another line, the tangent to our circle at its intersection with the interior of the line segment aA . This tangent hits AC and AB at points p and q , say.



Here, triangle Apq is again equilateral, with side lengths $1/3$. [Why?]

But the really important point concerns the next two pictures:



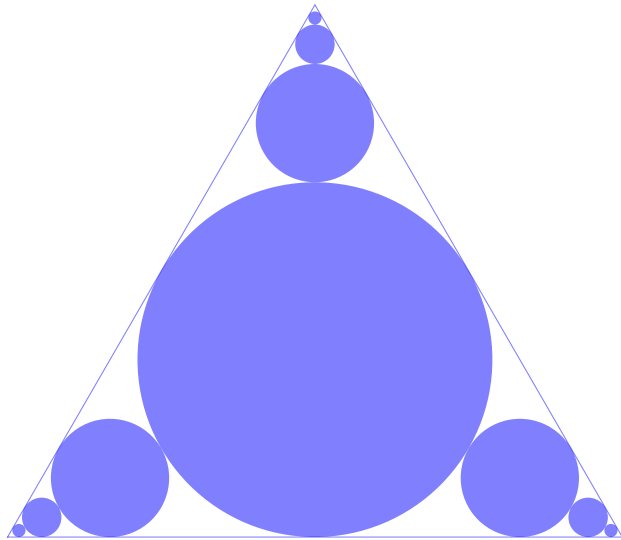
The whole batch of circles in the second picture is a scale model of the batch of circles in the first picture, with a scaling ratio of 1 to 3! Areas scale as the square of linear dimensions, so the area of all the circles in the second picture is one ninth the area of all the circles in the first picture.

Going over to algebra, if we say that the area of the circles in the second picture is T , then the area of all the circles in the first picture is $9T$. But it's also $T + \pi/12$. So

$$9T = T + \pi/12 \text{ So } 8T = \pi/12 \text{ and } T = \pi/96.$$

What about the original problem? Well, the areas in the original problem consist of the central circle of area $\pi/12$ together with three batches of smaller circles, each adding up to T . The total area then is $\pi/12 + 3T = \pi/12 + \pi/32 = 11\pi/96$.

Every answer, once computed, ought to be checked for plausibility. (Can Obama's trip to India etc. really be costing 200 million dollars a day?)



Just eyeballing this, what fraction of the area inside the triangle does the blue paint cover? Anybody?

```
In[68]:= N[Sqrt[3] / 4]
```

```
Out[68]= 0.433013
```

```
In[69]:= N[11 Pi / 96]
```

```
Out[69]= 0.359974
```

```
In[70]:= % / %%
```

```
Out[70]= 0.831325
```

Our computed answer, together with the easily computed area of the original triangle, says that the paint covers 83 percent of the canvas. That strikes me as about right, and I hope it doesn't outrage your artistic eye for area either.

Here's another problem in the realm of geometric series. Well, sort of: Let $X=1/1+2/2+3/4+4/8+5/16+6/32+7/64+8/128+9/256+\dots$. Find X .

That's not a geometric series, you say. Right you are, I say. But still, can we play this game? The game of introducing related things and comparing them to the original?

Let $Y=1+1/2+1/4+1/8+1/6+1/32+1/64+1/128+1/256+\dots$. Finding Y is easy enough; it fits the formula, or we could just use the underlying idea and note that $Y=1+Y/2$. Either way, $Y=2$.

Now $X-Y=$

```
1/1+2/2+3/4+4/8+5/16+6/32+7/64+8/128+9/256+...
minus
1/1+1/2+1/4+1/8+1/6+1/32+1/64+1/128+1/256+...
=
0/1+1/2+2/4+3/8+4/16+5/32+6/64+7/128+8/256+...
=
1/2+2/4+3/8+4/16+5/32+6/64+7/128+8/256+9/512+...
half of X, which was originally
1/1+2/2+3/4+4/8+5/16+6/32+7/64+8/128+9/256+... .
```

Summary: $X-2=X-Y=X/2$. After a bit of footwork, $X-X/2=2$, $X/2=2$, $X=4$.

Under the rug.

In case you ever want to do this yourself, but not from scratch.

I actually worked with a different scaling. The center is the origin $\{0,0\}$, and A is $\{0,2\}$.

In[14]:= ? **Disk**

```
Disk[{x, y}, r] is a two-dimensional graphics primitive
that represents a filled disk of radius r centered at the point x, y.
Disk[{x, y}] gives a disk of radius 1.
Disk[{x, y}, r, {θ1, θ2}] gives a segment of a disk.
Disk[{x, y}, {rx, ry}] gives an elliptical disk with semi-axes
of lengths rx and ry, oriented parallel to the coordinate axes. >>
```

```
In[16]:= trianglepicture[{ctr_, r_}] := Module[{a, b, c, d}, a = ctr + r {-Sqrt[3], -1};
b = ctr + r {Sqrt[3], -1}; c = ctr + r {0, 2}; d = Disk[ctr, r];
{Line[{a, b, c, a}], d}]
```

```
In[46]:= picture[{ctr_, r_}] := Module[{a, b, c, d}, a = ctr + r {-Sqrt[3], -1};
b = ctr + r {Sqrt[3], -1}; c = ctr + r {0, 2}; d = Disk[ctr, r];
{d}]
```

```
In[27]:= t0 = trianglepicture[{{0, 0}, 1}]
```

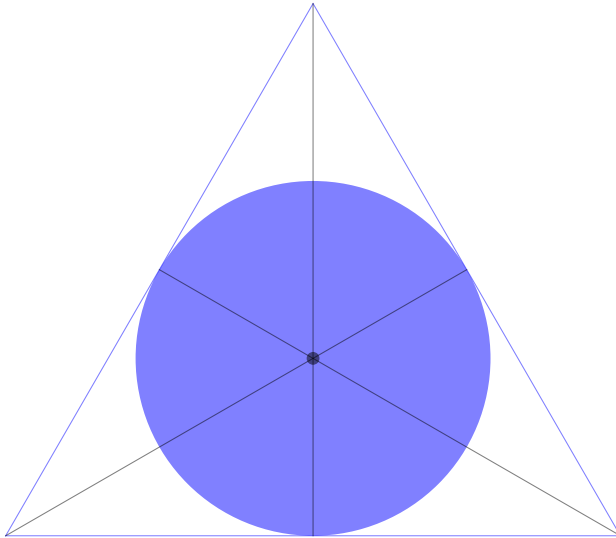
```
Out[27]= {Line[{{{-Sqrt[3], -1}, {Sqrt[3], -1}, {0, 2}, {-Sqrt[3], -1}}], Disk[{0, 0}, 1}]}
```

```
In[28]:= rad = Line[{{0, 0}, {0, -1}}]
```

```
Out[28]= Line[{{0, 0}, {0, -1}}]
```

```
In[33]:= Show[Graphics[Join[{Blue, Opacity[0.5]}, {t0, Black, Line[{{-Sqrt[3], -1}, {Sqrt[3], 1} / 2]},  
Line[{{0, -1}, {0, 2}}],  
Line[{{Sqrt[3], -1}, {-Sqrt[3], 1} / 2}], PointSize[0.02], Point[{0, 0}]}]]]
```

```
Out[33]=
```

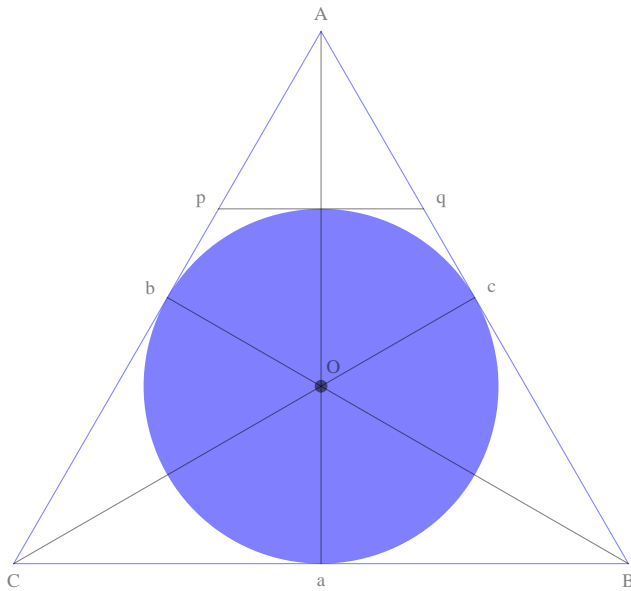


```

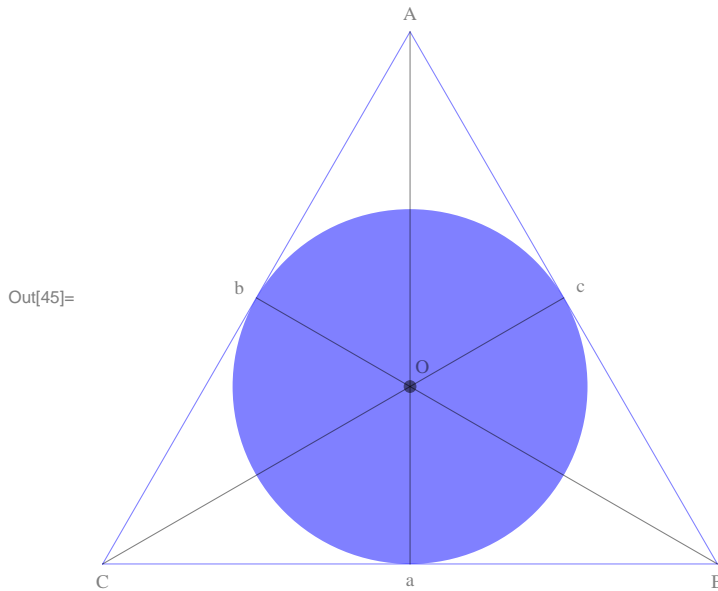
In[62]:= Show[Graphics[Join[{Blue, Opacity[0.5]}, {t0, Black, Line[{{-Sqrt[3], -1}, {Sqrt[3], 1} / 2}],
  Line[{{0, -1}, {0, 2}}],
  Line[{{Sqrt[3], -1}, {-Sqrt[3], 1} / 2}],
  Line[{p, q}], PointSize[0.02],
  Point[{0, 0}], Text["A", {0, 2.1}], Text["B", {Sqrt[3], -1.1}],
  Text["C", {-Sqrt[3], -1.1}],
  Text["a", {0, -1.1}],
  Text["b", {-Sqrt[3], 1} / 1.8],
  Text["c", {Sqrt[3], 1} / 1.8],
  Text["O", {0.07, 0.11}], Text["p", p + {-0.1, 0.05}],
  Text["q", q + {0.1, 0.05}]
}],
  ]]]]

```

Out[62]=




```
Show[Graphics[Join[{Blue, Opacity[0.5]}, {t0, Black, Line[{{-Sqrt[3], -1}, {Sqrt[3], 1} / 2}],
  Line[{{0, -1}, {0, 2}}],
  Line[{{Sqrt[3], -1}, {-Sqrt[3], 1} / 2}], PointSize[0.02],
  Point[{0, 0}], Text["A", {0, 2.1}], Text["B", {Sqrt[3], -1.1}],
  Text["C", {-Sqrt[3], -1.1}],
  Text["a", {0, -1.1}],
  Text["b", {-Sqrt[3], 1} / 1.8],
  Text["c", {Sqrt[3], 1} / 1.8],
  Text["O", {0.07, 0.11}]
}],
]]]]
```



```
In[19]:= extras[n_, s_] := Module[{ctr, rad}, ctr = (2 - 2 / 3^n) s; rad = 1 / 3^n;
  trianglepicture[{ctr, rad}]]
```

```
In[47]:= picture[{ctr_, r_}] := Module[{a, b, c, d}, a = ctr + r {-Sqrt[3], -1};
  b = ctr + r {Sqrt[3], -1}; c = ctr + r {0, 2}; d = Disk[ctr, r];
  {d}]
```

```
In[48]:= dextras[n_, s_] := Module[{ctr, rad}, ctr = (2 - 2 / 3^n) s; rad = 1 / 3^n;
  picture[{ctr, rad}]]
```

```
In[20]:= extras[1, {0, 1}]
```

```
Out[20]= {Line[{{{- 1 / Sqrt[3], 1}, {1 / Sqrt[3], 1}, {0, 2}, {- 1 / Sqrt[3], 1}}], Disk[{0, 4 / 3}, 1 / 3}]
```

```
In[21]:= extras[2, {0, 1}]
```

```
Out[21]= {Line[{{{- 1 / (3 Sqrt[3]), 5 / 3}, {1 / (3 Sqrt[3]), 5 / 3}, {0, 2}, {- 1 / (3 Sqrt[3]), 5 / 3}}], Disk[{0, 16 / 9}, 1 / 9}]
```

```
In[22]:= extras[1, {Sqrt[3] / 2, -1 / 2}]
```

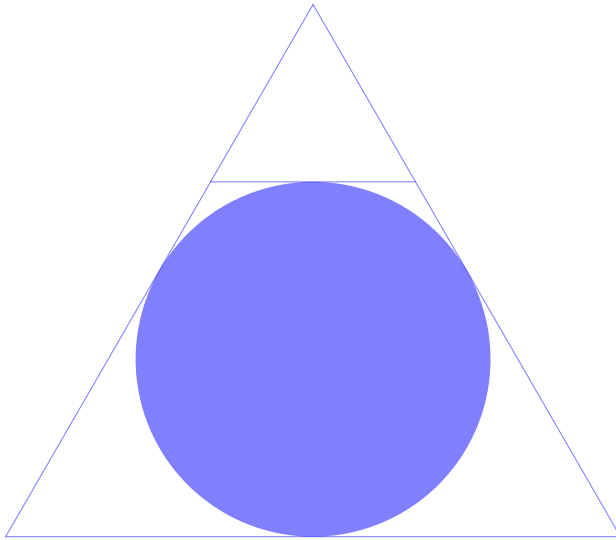
```
Out[22]= {Line[{{{1 / Sqrt[3], -1}, {Sqrt[3], -1}, {2 / Sqrt[3], 0}, {1 / Sqrt[3], -1}}], Disk[{2 / Sqrt[3], -2 / 3}, 1 / 3}]
```

```
In[59]:= p = {-Sqrt[3] / 3, 1}; q = {Sqrt[3] / 3, 1}
```

```
Out[59]=  $\left\{ \frac{1}{\sqrt{3}}, 1 \right\}$ 
```

```
In[60]:= Show[Graphics[Join[{Blue, Opacity[0.5], Line[{p, q]}], trianglepicture[{{0, 0}, 1}]]]
```

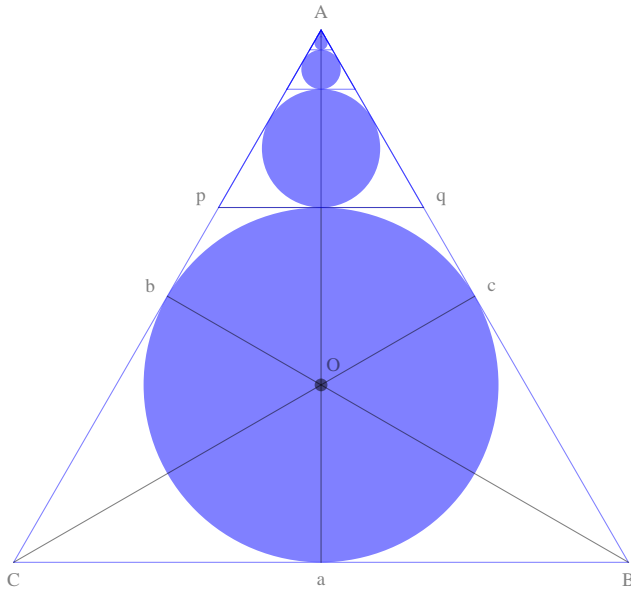
```
Out[60]=
```



```
In[66]:= Show[Graphics[Join[{Blue, Opacity[0.5]}, {t0, Black, Line[{{-Sqrt[3], -1}, {Sqrt[3], 1} / 2}],
  Line[{{0, -1}, {0, 2}}],
  Line[{{Sqrt[3], -1}, {-Sqrt[3], 1} / 2}],
  Line[{p, q}], PointSize[0.02],
  Point[{0, 0}], Text["A", {0, 2.1}], Text["B", {Sqrt[3], -1.1}],
  Text["C", {-Sqrt[3], -1.1}],
  Text["a", {0, -1.1}],
  Text["b", {-Sqrt[3], 1} / 1.8],
  Text["c", {Sqrt[3], 1} / 1.8],
  Text["O", {0.07, 0.11}], Text["p", p + {-0.1, 0.05}],
  Text["q", q + {0.1, 0.05}],
  Blue,
  extras[1, {0, 1}], extras[2, {0, 1}],
  extras[3, {0, 1}], extras[4, {0, 1}]

  ]]]]
```

Out[66]=



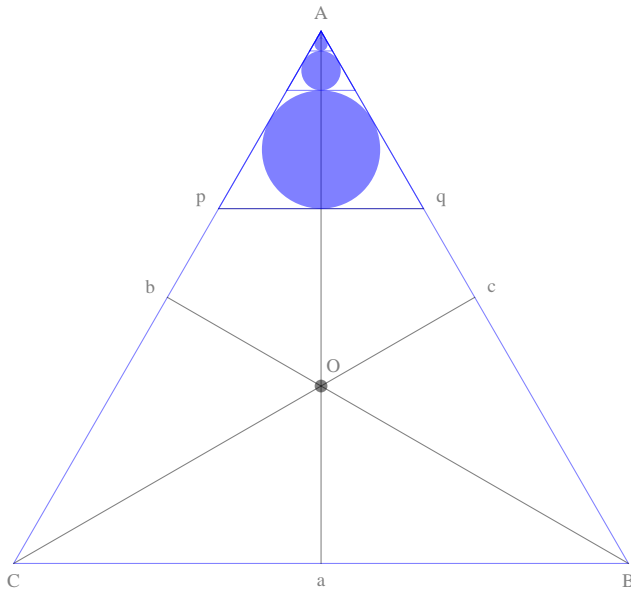
```

In[67]:= Show[
  Graphics[Join[{Blue, Opacity[0.5]}, {Line[{{0, 2}, {Sqrt[3], -1}, {-Sqrt[3], -1}, {0, 2}}],
    Black, Line[{{-Sqrt[3], -1}, {Sqrt[3], 1} / 2}],
    Line[{{0, -1}, {0, 2}}],
    Line[{{Sqrt[3], -1}, {-Sqrt[3], 1} / 2}],
    Line[{p, q}], PointSize[0.02],
    Point[{0, 0}], Text["A", {0, 2.1}], Text["B", {Sqrt[3], -1.1}],
    Text["C", {-Sqrt[3], -1.1}],
    Text["a", {0, -1.1}],
    Text["b", {-Sqrt[3], 1} / 1.8],
    Text["c", {Sqrt[3], 1} / 1.8],
    Text["O", {0.07, 0.11}], Text["p", p + {-0.1, 0.05}],
    Text["q", q + {0.1, 0.05}],
    Blue,
    extras[1, {0, 1}], extras[2, {0, 1}],
    extras[3, {0, 1}], extras[4, {0, 1}]

  ]]]]

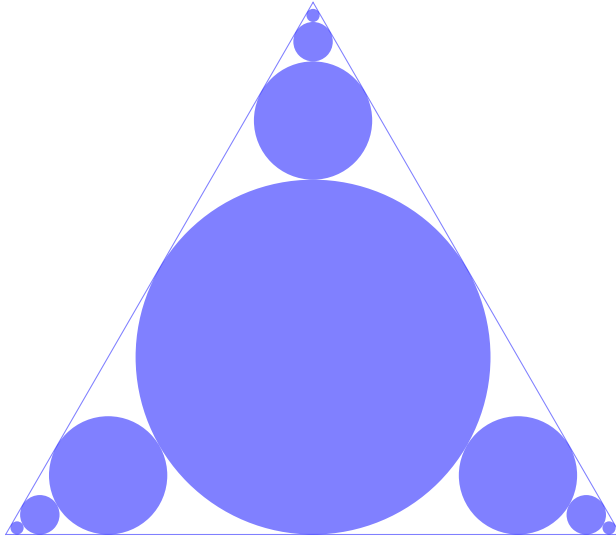
```

Out[67]=



```
In[50]:= Show[Graphics[Join[{Blue, Opacity[0.5]}, trianglepicture[{{0, 0}, 1]],
  dextras[1, {0, 1}], dextras[2, {0, 1}], dextras[3, {0, 1}],
  dextras[1, {Sqrt[3] / 2, -1 / 2}], dextras[2, {Sqrt[3] / 2, -1 / 2}],
  dextras[3, {Sqrt[3] / 2, -1 / 2}], dextras[1, {-Sqrt[3] / 2, -1 / 2}],
  dextras[2, {-Sqrt[3] / 2, -1 / 2}],
  dextras[3, {-Sqrt[3] / 2, -1 / 2}]
]]]
```

Out[50]=

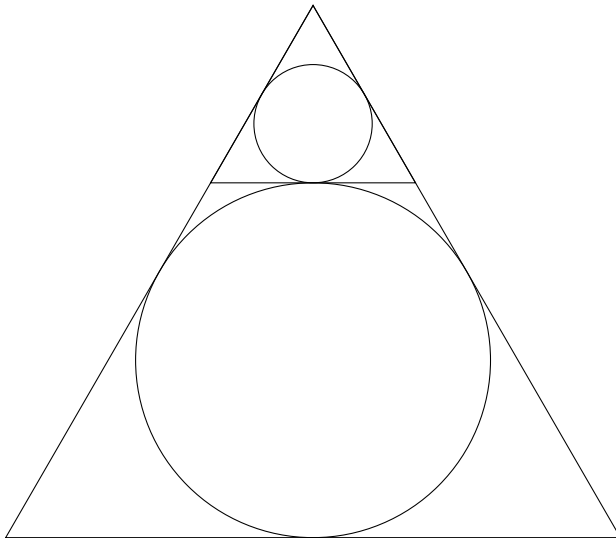


```
Export["triangles.pdf", %, "PDF"]
```

triangles.pdf

```
In[12]:= Show[Graphics[Join[extras[1, {0, 1}], extras[2, {0, 1}]]]]]
```

Out[12]=



```
In[13]:= extras[2, {0, 1}]
```

```
Out[13]= {Line[{{{-1/(3*sqrt(3)), 5/3}, {1/(3*sqrt(3)), 5/3}, {0, 2}, {-1/(3*sqrt(3)), 5/3}}}], Circle[{0, 16/9}, 1/9]}
```

```
In[24]:= Show[Graphics[Join[{Blue, Opacity[0.5]},  
  trianglepicture[{{0, 0}, 1}], extras[1, {0, 1}], extras[2, {0, 1}],  
  extras[3, {0, 1}], extras[1, {Sqrt[3] / 2, -1 / 2}], extras[2, {Sqrt[3] / 2, -1 / 2}],  
  extras[3, {Sqrt[3] / 2, -1 / 2}], extras[1, {-Sqrt[3] / 2, -1 / 2}],  
  extras[2, {-Sqrt[3] / 2, -1 / 2}],  
  extras[3, {-Sqrt[3] / 2, -1 / 2}]]
```