TAMU 2014 Freshman-Sophomore Math Contest

Sophomore Version

While the name of the contest is traditional, the actual eligibility rules are that first year students take the freshman contest, and second year students take the sophomore contest. That way, students who have accummulated enough credit hours in their first or second year to have standing as sophomores, or juniors, are not promoted out of eligibility.

The first page contains problems built around Calculus I and II for both freshmen and sophomores. The second pages are pitched to content unique to Calculus III and/or Differential Equations, in the case of the sophomore contest.

In all cases, solutions should be written out and should include reasoning behind the steps when reasons beyond routine calculation are involved. No tables, calculators, or computers, and no devices for communication with the outside world, are allowed. You're on your own.

1. Find
$$\int_{x=0}^{1} \frac{1}{x+\sqrt{x}}\,dx.$$
This one wasn't all that bad if you saw the substitution $u^2 = x$. The integral becomes
$$\int_{u=0}^{1} \frac{2u\,du}{u^2+u} = 2\int_{u=0}^{1} \frac{du}{u+1} = 2\int_{v=1}^{2} \frac{dv}{v} = 2\ln 2.$$

2. Find the coefficient of x^4 in the power series expansion (that is, the Taylor's series expansion about $a = 0$) of $e^{\sin x}$.
We have $e^u = 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \cdots$, and $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$. Putting these together by substitution gives
$$e^{\sin x} = 1 + \left(x - \tfrac{x^3}{6} + \tfrac{x^5}{120} - \cdots\right) + \tfrac{1}{2!}\left(x - \tfrac{x^3}{6} + \tfrac{x^5}{120} - \cdots\right)^2 + \cdots + \tfrac{1}{4!}\left(x - \tfrac{x^3}{6} + \tfrac{x^5}{120} - \cdots\right)^4 + \cdots.$$
The term 1 is not part of the answer. The next term contains all odd powers of x, so it's not part of the answer either. If you multiply out
$$\tfrac{1}{2!}\left(x - \tfrac{x^3}{6} + \tfrac{x^5}{120} - \cdots\right)^2$$
and extract the x^4 term, it works out to $-x^4/6$. The term involving u^3 again consists of odd powers of x, which leaves the u^4 term. It contributes $x^4/24$, so the coefficient on x^4 overall is
$$-\frac{1}{6} + \frac{1}{24} = -\frac{1}{8}$$

and that's the answer.

There's another approach involving taking derivatives. The coefficient is $1/4!$ times the 4th derivative evaluated at 0. Writing s for $\sin x$ and c for $\cos x$, the fourth derivative works out to $e^s(c^4 - 3c^3 + c^2)$ plus terms that involve s—and those terms are irrelevant to the evaluation at $x = 0$. Again we get $-1/8$.

3. Let

$$f(x) = \int_{t=1}^{2} \frac{1}{t + x^2 t^3}\, dt.$$

Find $f'(x)$ and, in particular, find the exact numerical value of $f'(x)$ at $x = 1$.

This integral can be hammered out, and it comes to an expression in x which can then be differentiated. But there's a trick solution as well. Write $u = xt$ and then $du = x\,dt$ and the original becomes

$$f(x) = \int_{u=x}^{2x} \frac{du}{u + u^3}.$$

By the fundamental theorem of calculus and the chain rule, the derivative with respect to x is thus

$$f'(x) = \frac{2}{2x + (2x)^3} - \frac{1}{x + x^3}.$$

Taking $x = 1$ gives $f'(1) = \frac{2}{10} - \frac{1}{2} = -\frac{3}{10}$.

4. Find the average value of the (unsigned) distance between two randomly placed points in the interval $[0, 1]$. Equivalently, find

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n} \sum_{k=1}^{n} \left| \frac{j}{n} - \frac{k}{n} \right|.$$

This limit is the defined meaning of $\int_{x=0}^{1} \int_{y=0}^{1} |x - y|\; dy\, dx$. That integral is the sum of the integral over the triangle where $y \ge x$ and the integral over the other triangle where $y \le x$. By symmetry, the sum is twice either of these.

In the lower triangle we have to evaluate

$$\int_{x=0}^{1} \int_{y=0}^{x} (x - y)\, dy\, dx = \int_{x=0}^{1} \tfrac{1}{2} x^2\, dx = \tfrac{1}{6}.$$

The answer is therefore that the average distance between two randomly chosen points in the unit interval is $1/3$.

5. Give an integral (single or multiple) that, if evaluated, would give the volume of the finite region bounded between the surface $z^2 = 4(x^2 + y^2)$ and the planes $z = x + 1$ and $z = 2/3$.

   Perhaps the easiest way to slice this thing is with horizontal slices. All slices will be portions of the disk cut from the cone of the problem by horizontal planes at elevation z, as z ranges from its minimum value of $2/3$ to its maximum value of 2. The cross sectional circle cut from the cone by a horizontal circle at elevation z has radius $z/2$.

   The bottom slice will be a disk of radius $1/3$ at elevation $2/3$. The top slice will be a single point $(1, 0, 2)$. In between, the slices of our solid at elevation z will be that portion of the disk $x^2 + y^2 \le (z/2)^2$ to the right of the plane perpendicular to the x-axis through $(z - 1, 0, z)$.

   So z ranges from $2/3$ to 2, x ranges from $z - 1$ to $z/2$, and y ranges over the interval

$$\left[-\sqrt{\left(\frac{z}{2}\right)^2 - x^2}, \sqrt{\left(\frac{z}{2}\right)^2 - x^2}\right].$$

   Thus an integral for the volume is

$$V = \int_{z=2/3}^{2} \int_{x=z-1}^{z/2} 2\sqrt{(z/2)^2 - x^2}\, dx\, dz.$$

   The inner integral expresses a geometry area problem and the answer works out to

$$A(z) = \left(\frac{3}{2}z - 1\right)\sqrt{(z/2)^2 - (z-1)^2} + \frac{1}{4}z^2 \cos^{-1}\left(2\left[1 - \frac{1}{z}\right]\right).$$

   So a single-variable integral that gives the required volume is $\int_{z=2/3}^{2} A(z)\, dz$. Perhaps surprisingly (it's a scary-looking formula) or perhaps not (it's a natural geometric question) this integral turns out to be one of those that can be evaluated in closed form. The final evaluation, courtesy of Mathematica, is $V = \frac{2\pi}{81}(3\sqrt{3} - 1)$.

6. Water flows out of a hole at the bottom of a spherical bowl of radius 1 meter. (Air flows in through a hole at the top.) The bowl starts out full of water, but after 10 minutes, it's half empty. The outflow runs at a rate proportional to the depth of the remaining water.

   (a) Set up a differential equation for the relation between the depth h of the remaining water and the time t since the bowl began to empty out.

   It is convenient to introduce the additional variables z and V for the distance the water level lies above the half-empty mark, and the remaining volume in the bowl, respectively. Note that $h = 1 + z$. It is given in the problem that $dV/dt = -Kh = -K(1 + z)$, where K

is an as-yet undetermined positive constant. We also have $dV/dz = \pi(1-z^2)$, because that's the area of the cross section of the sphere at elevation z. Now

$$\frac{dz}{dt} \cdot \frac{dV}{dz} = \frac{dV}{dt}$$

so

$$\frac{dz}{dt} \cdot \pi(1-z^2) = -K(1+z).$$

Solving for dz/dt and substituting $z = h-1$ in here, and noting that $dh/dt = dz/dt$, we have

$$\frac{dh}{dt} = -\frac{Kh}{\pi(1-(h-1)^2)}.$$

(b) How long will it be before the bowl is empty, counting from when the bowl was full?

It's probably better to work this in terms of z: $z = 1$ and $t = 0$ at the outset. $z = -1$ at the end...and what is t? Our differential equation becomes

$$\frac{dz}{dt} = -\frac{K(1+z)}{\pi(1-z^2)} = -\frac{K}{\pi(1-z)}.$$

So

$$\frac{dt}{dz} = -\frac{\pi(1-z)}{K}.$$

Thus

$$t = \tfrac{K}{2\pi}(1-z)^2 + J$$

where J and K are constants that must be chosen to fit the data. When $z = 1$, $t = 0$ which forces $J = 0$. (Lucky us! We could have forgotten about J and got away with it.) When $z = 0$, $t = 10$. So $10 = K/(2\pi)$ and $K = 20\pi$. Finally, taking $z = -1$ gives $t = 10(1-z)^2 = 40$. So the answer is that starting from a full bowl it takes 40 minutes to get to empty.