

# TAMU 2014 Freshman-Sophomore Math Contest

## Freshman Version

While the name of the contest is traditional, the actual eligibility rules are that first year students take the freshman contest, and second year students take the sophomore contest. That way, students who have accumulated enough credit hours in their first or second year to have standing as sophomores, or juniors, are not promoted out of eligibility.

The first page contains problems built around Calculus I and II for both freshmen and sophomores. The second pages are pitched to content unique to Calculus III and/or Differential Equations, in the case of the sophomore contest.

In all cases, solutions should be written out and should include reasoning behind the steps when reasons beyond routine calculation are involved. No tables, calculators, or computers, and no devices for communication with the outside world, are allowed. You're on your own.

1. Find

$$\int_{x=0}^1 \frac{1}{x + \sqrt{x}} dx.$$

This one wasn't all that bad if you saw the substitution  $u^2 = x$ . The integral becomes

$$\int_{u=0}^1 \frac{2u du}{u^2 + u} = 2 \int_{u=0}^1 \frac{du}{u + 1} = 2 \int_{v=1}^2 \frac{dv}{v} = 2 \ln 2.$$

2. Find the coefficient of  $x^4$  in the power series expansion (that is, the Taylor's series expansion about  $a = 0$ ) of  $e^{\sin x}$ .

We have  $e^u = 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \frac{1}{4!}u^4 + \dots$ , and  $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots$ . Putting these together by substitution gives

$$\begin{aligned} e^{\sin x} &= 1 + \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right) + \frac{1}{2!} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)^2 \\ &\quad + \dots + \frac{1}{4!} \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)^4 + \dots \end{aligned}$$

The term 1 is not part of the answer. The next term contains all odd powers of  $x$ , so it's not part of the answer either. If you multiply out

$$\frac{1}{2!} \left( x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \right)^2$$

and extract the  $x^4$  term, it works out to  $-x^4/6$ . The term involving  $u^3$  again consists of odd powers of  $x$ , which leaves the  $u^4$  term. It contributes  $x^4/24$ , so the coefficient on  $x^4$  overall is

$$-\frac{1}{6} + \frac{1}{24} = -\frac{1}{8}$$

and that's the answer.

There's another approach involving taking derivatives. The coefficient is  $1/4!$  times the 4th derivative evaluated at 0. Writing  $s$  for  $\sin x$  and  $c$  for  $\cos x$ , the fourth derivative works out to  $e^s(c^4 - 3c^3 + c^2)$  plus terms that involve  $s$ —and those terms are irrelevant to the evaluation at  $x = 0$ . Again we get  $-1/8$ .

3. Let

$$f(x) = \int_{t=1}^2 \frac{1}{t + x^2 t^3} dt.$$

Find  $f'(x)$  and, in particular, find the exact numerical value of  $f'(x)$  at  $x = 1$ .

This integral can be hammered out, and it comes to an expression in  $x$  which can then be differentiated. But there's a trick solution as well. Write  $u = xt$  and then  $du = x dt$  and the original becomes

$$f(x) = \int_{u=x}^{2x} \frac{du}{u + u^3}.$$

By the fundamental theorem of calculus and the chain rule, the derivative with respect to  $x$  is thus

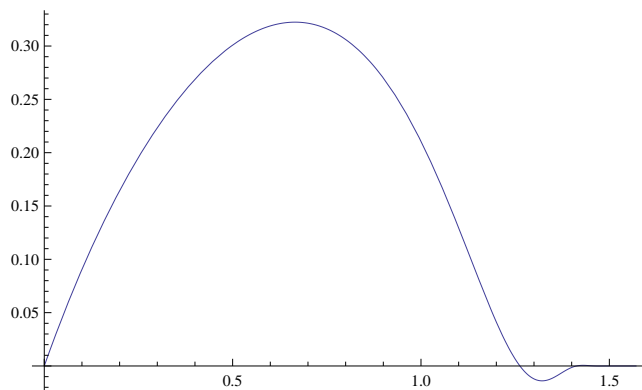
$$f'(x) = \frac{2}{2x + (2x)^3} - \frac{1}{x + x^3}.$$

Taking  $x = 1$  gives  $f'(1) = \frac{2}{10} - \frac{1}{2} = -\frac{3}{10}$ .

4. Sketch the graph of

$$g(x) = e^{-\tan x} \sin(\tan x)$$

on the interval  $0 \leq x \leq \pi/2$ , and find the maximum value of  $g(x)$  on the interval, as well as the place or places where that maximum is achieved. The technically accurate figure, thanks to a computer algebra system, looks like this:



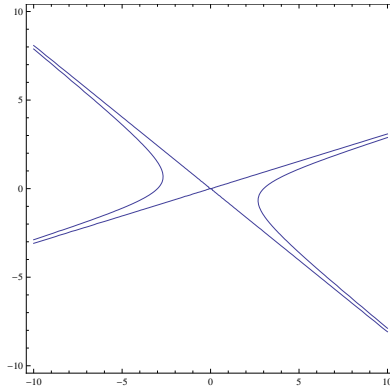
but that's not quite the whole of the story. Submerged beneath the fact that the exponential is tailing off so fast is the fact that the function in question is, on a small scale, wiggling ever faster. So it goes back and forth between positive and negative infinitely many times, only, back and forth only a little.

As to the maximum value, as  $x$  ranges from 0 to  $\pi/2$ ,  $\tan x$  ranges from 0 to  $\infty$ . So what we're really looking for is the maximum value of  $e^{-u} \sin u$  over  $u \geq 0$ .

The maximum will not be reached at either end, because at 0 the function is 0 and as  $u$  tends to  $\infty$  again it goes to 0. So it must occur at some point in the middle where the derivative is zero. That happens exactly when  $\sin u = \cos u$ , and that happens at  $\pi/4 + n\pi$ . The first such  $u$  is best because for larger  $u$ , the exponential factor hurts. When  $u = \tan x = \pi/4$ ,  $x = \tan^{-1} \pi/4$  and that's the answer.

5. The graph of  $x^2 - 2xy - 4y^2 = 9$  is a hyperbola.

- (a) Sketch the graph and find the equations of its asymptotes.



The asymptotes are given by  $x^2 - 2xy - 4y^2 = 0$ . There are two lines. The lines are

$$y = \frac{-1 \pm \sqrt{5}}{4}x.$$

- (b) Find a point on the curve that is nearest the  $y$ -axis.

We are trying to minimize  $x$  subject to  $x > 0$  and  $x^2 - 2xy - 4y^2 = 9$ . The minimum will occur when  $dy/dx = \infty$  or when  $dx/dy = 0$ . Differentiating the equation with respect to  $y$  and thinking of  $x$  as a function of  $y$ , we get  $dx/dy = (x + 4y)/(x - y)$ . This is zero when  $x = -4y$ . So we have to solve  $x = -4y$  together with  $x > 0$  and  $x^2 - 2xy - 4y^2 = 9$ . That gives  $y = \pm\sqrt{9/20}$  and so our answer is  $x = 4\sqrt{9/20} = 6/\sqrt{5}$ .

6. The radius of the earth is (about, but please use this figure)  $5 * 10^6$  meters. A cable of unobtainium massing  $1\text{kg}/\text{meter}$  is hung from a tower (built of the same stuff)  $10^7$  meters tall. For your convenience, the earth has been made to stop spinning for the duration of this problem. (If your head is spinning, that's your problem.) The acceleration due to gravity at the earth's surface is (about, but please use this figure)  $10\text{m}/\text{s}^2$ . Work is measured in joules. (That is,  $1\text{j} = 1\text{kgm}^2/\text{s}^2$ . One joule is the energy needed to accelerate  $1\text{ kg}$  to a speed of  $1\text{ meter per second}$ .)

How much work would it take to winch up the whole cable to the top of the tower? Take into account that the force of gravity is inversely proportional to the square of the distance from the center of the earth.

Let's take  $x$  to be the distance to the center of the earth from wherever we're talking about. The base of the tower has elevation  $5 * 10^6$ . The top has elevation  $15 * 10^6$ . A meter of cable at elevation  $s$  (measuring from the center) weighs  $10(5 * 10^6)^2 s^{-2}$  newtons and costs that many joules to lift one meter.

The cable, once it's been winched up to where its lower end is  $x$  meters from the center, weighs  $F(x) = \int_{s=x}^{15*10^6} 10(5 * 10^6)^2 s^{-2} ds$  newtons, in the sense that that's how much force is needed to keep it going up as you winch. This integral, apart from the messy constants, is simple enough and it evaluates to

$$F(x) = 5 * 10^{13} \left( \frac{1}{x} - \frac{1}{15 * 10^6} \right).$$

The total work needed, in joules, is thus

$$\int_{x=5*10^6}^{15*10^6} F(x) dx = 5 * 10^{13} \left( \ln 3 - \frac{2}{3} \right).$$

Remark: serious thought has been given to building very large cables. No one sees any hope of building a device of the sort described here, but a cable stretching from the surface of a miniature planet or small moon that was spinning at a convenient rate might be doable at some point in the distant future. It'd have to vary in thickness, and the challenges of computing its behavior would be considerable.