Solutions 2010 TAMU Freshman-Sophomore Math Contest
Second-year student version

1. Find
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}.$$
This is a telescoping series. $1/((n+1)(n+2)) = 1/(n+1) - 1/(n+2)$, so $1/(n(n+1)(n+2)) = 1/n(1/(n+1) - 1/(n+2)) = 1/n - 1/(n+1) - (1/2)((1/n) - 1/(n+2))$. Now summing the first piece of this gives $1/1$, while with the second piece, the first two terms survive untelescoped so we have a contribution of $-(1/2)(1+1/2) = -3/4$. Since $1 - 3/4 = 1/4$, the answer to the question is $1/4$.

2. Lake Cony has a radius of 1000 meters and fills a conical depression 100 meters deep. Water masses 1000 kg per cubic meter, and the acceleration due to gravity is 9.81 meters/second^2. A joule is the energy needed to accelerate a mass of 1 kilogram to a speed of 1 meter per second. Find the energy (expressed in joules) needed to pump lake Cony dry.

   This is a method-of-disks integral problem. The disk that is x meters off the bottom of the lake has a radius of $10x$ meters, and an area of $100\pi x^2$. It must be raised $(100-x)$ meters. Thus we have $\int_{x=0}^{100} 100\pi x^2(100-x)\,dx$ cubic-meter-meter lifts of work to do. To lift one cubic meter of water one meter is to move 1000 kgs with a force of 9.81 newtons each, up one meter, and that takes 9810 joules. Grinding out the details gives $8.175\pi E12$ joules. (1.6 m kwH)

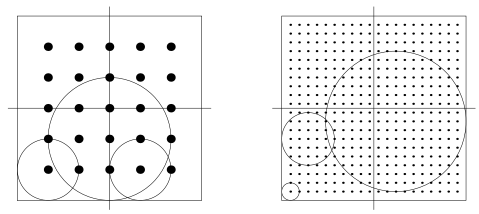
3. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$, where $a_0 = 1$, $a_1 = 1/2$, $a_2 = -1/8$, $a_3 = 1/16$, $a_4 = -5/128$, and in general, for $n \geq 1$, $a_n = -(n-3/2)a_{n-1}/n$.

   (a) Find a_5. The rule specifying the general case gives $a_5 = -(5 - 3/2)a_4/5$, and $a_4 = -5/128$, so $a_5 = -(5-3/2)(-5/128)/5 = 7/256$.

   (b) Multiply out $f(x) \cdot f(x)$ at least to the x^4 term and then take an informed guess at a simple formula for $f(x)^2$. This would amount to expanding $(1 + x/2 - (1/8)x^2 + (1/16)x^3 - 5/128x^4 + \cdots)^2$ and this multiplies out to $1 + x + 0x^2 + 0x^3 + 0x^4 + ?x^5 + \cdots$. (The coefficient on x^4 in the expansion is $2 * (-5/128 + (1/16) * 1/2) + (-1/8)^2 = 0$, and the others are easier.) Guess: $f(x)^2 = 1 + x$.

   (c) Prove your guess. The series for $f(x)$ is the Taylor's series expansion for $(1+x)^{1/2}$ about $x = 0$ because the nth derivative at zero of $(1+x)^{1/2}$ is the product of $(1/2 - j)$ over j from 0 to $n-1$, and that's equivalent to the product of $(3/2 - k)$ over k from 1 to n, and then we have to divide by $n!$ to get the coefficient in the Taylor's series. This product obeys exactly the recursive rule given in the problem, relating a_n to a_{n-1}, because to extend the product by one step from $n-1$ to n we multiply by $n-3/2$ and then the $n!$ brings in a factor of $1/n$.

4. An $n \times n$ box of points is set in a square formation inside the square $-1 \le x \le 1$, $-1 \le y \le 1$. Around each point, a circle is drawn, reaching just to the perimeter of the square. The figures show the cases $n = 5$ and $n = 20$, with just a few of the n^2 circles drawn in.



   (a) Find the average value of the area of the 25 circles associated with the case $n = 5$. Going by the picture, the distance from one point to the next, and from the outermost points to the edge of the square, is $1/3$. There are sixteen dots at a distance of $1/3$ from the edge, 8 squares at a distance $2/3$, and one at distance 1. The sum of the areas is thus $\pi(1/9)(16 * 1 + 8 * 4 + 1 * 9) = 19\pi/3$ and the average area is $19\pi/75$.

   (b) Find the limit, as n tends to infinity, of the average value of the area of the the n^2 circles drawn about the $n \times n$ points in the nth figure. This becomes a double integral. The average over the entire square array is equal to the average over any typical sector.
   We take the sector $0 \le x \le 1$, $-x \le y \le x$. This sector has area 1, so the average value of the area of a circle will be the integral of the area of the circle centered at (x, y). That is, our answer is

$$\int_{x=0}^{1} \int_{y=-x}^{x} \pi(1-x)^2 \, dy \, dx.$$

   The inner integral works out to $2\pi x(1-x)^2$, and integrating this from 0 to 1 gives the final answer, $\pi/6$. A related fact is that the average square of the distance to the edge, from a random point inside this square, is $1/6$.

5. Let $f(x, y) = 2x^3 - xy^2 + y$. Find the critical points of f and characterize them as local maxima, local minima, or saddle. Set both partials to zero to arrive at $x = \pm 24^{-1/4}$, $y = \pm(3/2)^{1/4}$. What sort of critical point? The determinant of the matrix of second partials holds the answer. Here, we have

$$\det \begin{pmatrix} 12x & -2y \\ -2y & -2x \end{pmatrix} = -24x^2 - 4y^2$$

which is negative at both extreme points. Thus both critical points are saddle points.