2006 Texas A&M University Freshman-Sophomore Math Contest
Second-year bracket version

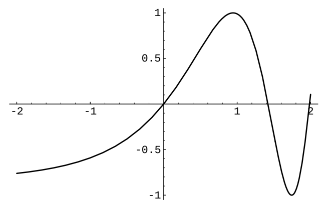
All the necessary steps and supporting calculations must be shown to earn full credit on a problem. There are six problems. The first three are common to both the first-year contestant version and the second-year contestant version. The second three branch out to cover different topics depending which bracket you're in.

No calculators or other electronic devices are allowed. Please turn cell phones off—the racket of a ring tone can be most distracting.

1. Find
$$\int_0^1 x\left(\ln(x+1)-\ln(x)\right)\,dx.$$
There are a couple of ways to work this. First, integration by parts. Taking $U=\ln(x+1)-\ln(x)$ and $dV = x\,dx$ gives
$$\tfrac{1}{2}x^2(\ln(x+1)-\ln(x))\,|_0^1 - \tfrac{1}{2}\int_0^1 x^2\left(\frac{1}{x+1}-\frac{1}{x}\right)\,dx.$$
The nonintegral term here evaluates to $\frac{1}{2}\ln(2)$, the easy integral evaluates to $1/4$, and the not quite so easy integral can be rewritten as
$$-\tfrac{1}{2}\int_0^1 (x^2+x-x-1+1)/(1+x)\,dx = \tfrac{1}{4}-\tfrac{1}{2}\ln(2).$$
Adding the pieces gives $1/2$ which is the answer. Another approach is to rewrite $\ln(x+1)-\ln(x)$ as $\int_0^1 \frac{1}{x+y}\,dy$ so that the whole integral becomes
$$\int_{x=0}^1\int_{y=0}^1 \frac{x}{x+y}\,dy\,dx.$$
Now by symmetry, if we switch the rôles of x and y in this double integral we get the same answer, but $\int\int(x+y)/(x+y)\,dy\,dx = 1$ over the square region of integration, so twice the answer is 1, and thus the answer itself is again $1/2$.

2. Consider the curve $g(x)=\sin(e^x-1)$.

   (a) Sketch the graph of $g(x)$ for $-2\le x\le 2$ and find the local maxima and minima. We have $g'(x)=e^x\cos(e^x-1)$ and the maxima and minima occur at places where this is zero. The second derivative is $e^x(\cos(e^x-1)-\sin(e^x-1))$ which is never zero when the first derivative is zero. Now $\cos(u)$ takes the value 0 when u is an odd multiple of $\pi/2$, so the maxima are the places where $e^x-1=\pi/2\pm 2k\pi$ and the minima are the places where $e^x-1=-\pi/2\pm 2k\pi$. There is one local maximum in the interval, and it is at $x=\ln(1+\pi/2)$. Two possible local minima are at $x = \ln(1-\pi/2)$ and at $x = \ln(1+3\pi/2)$. It works out that the first candidate is ineligible because negative numbers do not have logs, but the second is less than 2 because $1 + 3\pi/2 < e^2$. The curve is a kind of sine wave but with the wavelength changing, becoming shorter as x increases. (A smaller increase in x will advance $e^x - 1$ from one crest at $\pi/2 + 2\pi k$ to the next as x increases).



(b) Find the first four coefficients $a_1, \ldots, a_4$ in the power series expansion

$$g(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots .$$

There are a couple of ways to go at this. One would be to take derivatives, first through fourth, then set x to zero in each expression, evaluate the expression, and divide by $k!$ to get the coefficients. The other would be to use the series expansions of sin and of the exponential function, if you know them, and grind out the algebra:

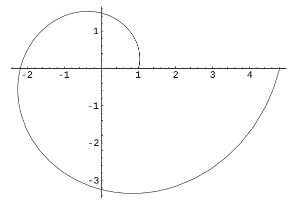
$$\begin{aligned}
\sin(u) &= u - u^3/3! + O(u^5); \\
e^x - 1 &= x + x^2/2! + x^3/3! + x^4/4! + O(x^5) \Rightarrow \\
\sin(e^x - 1) &= (x + x^2/2! + x^3/3! + x^4/4!) \\
&\quad - (x + x^2/2! + x^3/3! + x^4/4!)^3/3! + O(x^5) \\
&= (x + x^2/2 + x^3/6 + x^4/24) - (x + x^2/2)^3/6 + O(x^5) \\
&= (x + x^2/2 + x^3/6 + x^4/24) - (x^3 + 3x^4/2)/6 + O(x^5) \\
&= x + x^2/2 - (5/24)x^4 + O(x^5).
\end{aligned}$$

3. Find

$$\lim_{h\to 0} \frac{1}{h} \left(\lim_{t\to 0} \frac{1}{t} \left(\sin(x+h+t) - \sin(x+h) - \sin(x+t) + \sin x\right) \right).$$

Unwrapping the definitions, the inner limit is $\cos(x+h) - \cos(x)$ and the outer limit is thus $-\sin x$.

4. An ant crawls at a uniform speed along the spiral with parametric equation



$$x = e^{t/4} \cos t, \qquad y = e^{t/4} \sin t :$$

The ant starts at $(1, 0)$, corresponding to $t = 0$, and ends at $(e^{\pi/2}, 0)$, corresponding to $t = 2\pi$.

(a) How far did the ant crawl? The integral for arc length of a curve given by a parametric equation is $\int \sqrt{x'^2 + y'^2}\, dt$. Here, this gives

$$\int_0^{2\pi} \sqrt{17/16} e^{t/4}\, dt = \sqrt{17}(e^{\pi/2} - 1).$$

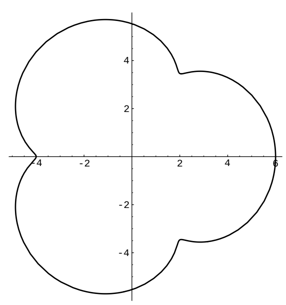
(b) What was the average straight line distance from where the ant was, along its trip, to the origin? For this one, we cannot simply integrate the distance dt and be done with it, because the ant's speed is uniform so it takes more time to traverse the longer arcs corresponding to intervals $(t, t + \Delta t)$ when t is larger. Instead, we must weight these arcs according to how long they are. And then, to get the average, divide by the total length L computed in the first part. This gives

$$\frac{1}{L} \int_0^{2\pi} e^{t/4} \frac{\sqrt{17}}{4} e^{t/4}\, dt = \frac{e^{\pi} - 1}{2(e^{\pi/2} - 1)} = \frac{1}{2}\left(e^{\pi/2} + 1\right).$$

5. Consider the curve with parametric equations

$$x = 5 \cos t + \cos 4t,$$
$$y = 5 \sin t + \sin 4t.$$

(a) Sketch the curve.



(b) Using Green's theorem, or in some other fashion, find the area of the region enclosed by the curve. In Green's theorem, we have

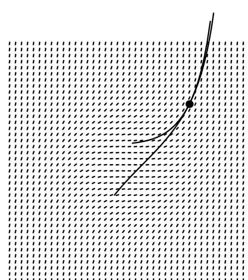
$$\int\int_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy\, dx = \oint_C (M\, dx + N\, dy).$$

Taking $M = 0$ and $N = x$ serves to make the LHS of Green's theorem an integral of 1 over the region enclosed by the curve. Evaluating the RHS gives

$$\oint_C x\,dy = \int_{t=0}^{2\pi} (5\cos t + \cos 4t)(5\cos t + 4\cos 4t)\,dt = 29\pi.$$

6. Consider the differential equation $y' = x^2 + y^2$.

   (a) Sketch a direction field for the solutions of the differential equation. We show the direction field together with a solution to the given equation and a solution to a related equation. The top curve is the solution to the given equation. The lower curve is a solution to the related differential equation $y' = 1+y^2$. For $x > 1$, the solution to this related curve will lie below the solution to the given curve, because the derivative $1+y^2$ will be less than the derivative for the other one, x^2+y^2 for two reasons: First, that $1 < x^2$, and second, that the value of y for the second curve is itself greater than for the first curve. What happens for $x < 1$ is irrelevant to the problem, but for $0 < x < 1$, the solution to the modified differential equation has a greater slope than does the solution to the given equation because $1+y^2 > x^2+y^2$ and because of the compounding effect of having a greater y value for the given differential equation than for the modified one. The solution to $y' = 1 + y^2$ is simple though, $x = \arctan(y) + C$. Fitting C to the initial point through which the solution must run gives $x = \arctan(y) + (1 - \pi/4)$. But then as x approaches $1 + \pi/4, 2$, y runs off to infinity. With the solution to the actual differential equation, y escapes to infinity sooner. This solves the first and third part.



   (b) For the particular solution $y = u(x)$ through $(0, 0)$, find $u'(0)$, $u''(0)$, and $u'''(0)$.These derivatives are 2 and 6 respectively. The first is simple: $y' = x^2 + y^2$ and at $x = y = 1$, this evaluates to 2. For the second, we have $y'' = (x^2+y(x)^2)' = 2x+2y(x)y'(x)$ which evaluated at $x = 1$ gives $2 + 4 = 6$.

   (c) For the particular solution $y = v(x)$ through $(1, 1)$, show that y escapes to infinity before x reaches 2. That is, there is no point of the form $(2, t)$ on the solution $y = v(x)$.