2006 Texas A&M University Freshman-Sophomore Math Contest
Freshman version

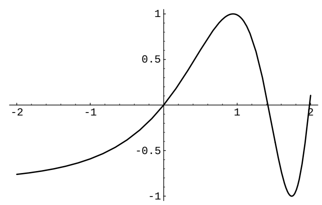
All the necessary steps and supporting calculations must be shown to earn full credit on a problem. There are six problems. The first three are common to both the first-year contestant version and the second-year contestant version. The second three branch out to cover different topics depending which bracket you're in.

No calculators or other electronic devices are allowed. Please turn cell phones off—the racket of a ring tone can be most distracting.

1. Find
$$\int_0^1 x\,(\ln(x+1)-\ln(x))\;dx.$$
There are a couple of ways to work this. First, integration by parts. Taking $U=\ln(x+1)-\ln(x)$ and $dV = x\,dx$ gives
$$\tfrac{1}{2}x^2(\ln(x+1)-\ln(x))\,|_0^1 - \tfrac{1}{2}\int_0^1 x^2\left(\frac{1}{x+1}-\frac{1}{x}\right)\,dx.$$
The nonintegral term here evaluates to $\frac{1}{2}\ln(2)$, the easy integral evaluates to $1/4$, and the not quite so easy integral can be rewritten as
$$-\tfrac{1}{2}\int_0^1 (x^2+x-x-1+1)/(1+x)\,dx = \tfrac{1}{4}-\tfrac{1}{2}\ln(2).$$
Adding the pieces gives $1/2$ which is the answer. Another approach is to rewrite $\ln(x+1)-\ln(x)$ as $\int_0^1 \frac{1}{x+y}\,dy$ so that the whole integral becomes
$$\int_{x=0}^1\int_{y=0}^1 \frac{x}{x+y}\,dy\,dx.$$
Now by symmetry, if we switch the rôles of x and y in this double integral we get the same answer, but $\int\int(x+y)/(x+y)\,dy\,dx = 1$ over the square region of integration, so twice the answer is 1, and thus the answer itself is again $1/2$.

2. Consider the curve $g(x) = \sin(e^x-1)$.

   (a) Sketch the graph of $g(x)$ for $-2 \le x \le 2$ and find the local maxima and minima. We have $g'(x) = e^x\cos(e^x-1)$ and the maxima and minima occur at places where this is zero. The second derivative is $e^x(\cos(e^x-1)-\sin(e^x-1))$ which is never zero when the first derivative is zero. Now $\cos(u)$ takes the value 0 when u is an odd multiple of $\pi/2$, so the maxima are the places where $e^x-1 = \pi/2\pm 2k\pi$ and the minima are the places where $e^x-1 = -\pi/2\pm 2k\pi$. There is one local maximum in the interval, and it is at $x=\ln(1+\pi/2)$. Two possible local minima are at $x = \ln(1-\pi/2)$ and at $x = \ln(1+3\pi/2)$. It works out that the first candidate is ineligible because negative numbers do not have logs, but the second is less than 2 because $1 + 3\pi/2 < e^2$. The curve is a kind of sine wave but with the wavelength changing, becoming shorter as x increases. (A smaller increase in x will advance $e^x - 1$ from one crest at $\pi/2 + 2\pi k$ to the next as x increases).



(b) Find the first four coefficients $a_1, \ldots, a_4$ in the power series expansion

$$g(x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots .$$

There are a couple of ways to go at this. One would be to take derivatives, first through fourth, then set x to zero in each expression, evaluate the expression, and divide by $k!$ to get the coefficients. The other would be to use the series expansions of sin and of the exponential function, if you know them, and grind out the algebra:

$$\begin{aligned}
\sin(u) &= u - u^3/3! + O(u^5); \\
e^x - 1 &= x + x^2/2! + x^3/3! + x^4/4! + O(x^5) \Rightarrow \\
\sin(e^x - 1) &= (x + x^2/2! + x^3/3! + x^4/4!) \\
&\quad - (x + x^2/2! + x^3/3! + x^4/4!)^3/3! + O(x^5) \\
&= (x + x^2/2 + x^3/6 + x^4/24) - (x + x^2/2)^3/6 + O(x^5) \\
&= (x + x^2/2 + x^3/6 + x^4/24) - (x^3 + 3x^4/2)/6 + O(x^5) \\
&= x + x^2/2 - (5/24)x^4 + O(x^5).
\end{aligned}$$

3. Find

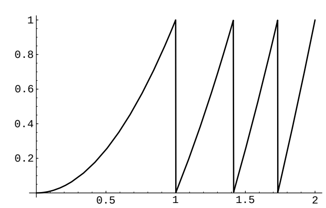
$$\lim_{h\to 0} \frac{1}{h} \left(\lim_{t\to 0} \frac{1}{t} \left(\sin(x+h+t) - \sin(x+h) - \sin(x+t) + \sin x\right) \right).$$

Unwrapping the definitions, the inner limit is $\cos(x+h) - \cos(x)$ and the outer limit is thus $-\sin x$.

4. Find
$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)(n+3)}.$$
This is a job for partial fractions. Recall that
$$\frac{1}{u(u+a)} = \frac{1}{a}\left(\frac{1}{u} - \frac{1}{u+a}\right).$$
Using this twice gives
$$\begin{aligned}\frac{1}{(n+1)(n+2)(n+3)} &= \frac{1}{n+1}\left(\frac{1}{n+2} - \frac{1}{n+3}\right)\\ &= \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+3} - \frac{1}{n+2} + \frac{1}{n+4}\right).\end{aligned}$$
Rearranging the contents of the big parentheses gives two telescoping sums, one in which the leading, and only surviving, term is 1/2, and the other, in which the surviving term is -1/3. Half the total is 1/12 which is the answer.

5. Let $\lfloor u \rfloor$ denote the integer part of u, so that $\lfloor 3.14159 \rfloor = 0.14159$, for instance. Let $f(x) = x^2 - \lfloor x^2 \rfloor$.

   (a) Graph $f(x)$ for $0 \le x \le 2$.



   (b) Show that for $\sqrt{n} \le x < \sqrt{n+1}$,
$$2\sqrt{n}(x - \sqrt{n}) \le f(x) \le (\sqrt{n} + \sqrt{n+1})(x - \sqrt{n}).$$
This holds because
$$x^2 - n = (x - \sqrt{n})(x + \sqrt{n})$$
and $(x + \sqrt{n})$ lies between $2\sqrt{n}$ and $\sqrt{n} + \sqrt{n+1}$.

   (c) Find
$$\lim_{N\to\infty} \frac{1}{N} \int_0^N f(x)\,dx.$$
As your composer of problems and solutions has grown weary of elaborate and detailed reasons, here is a simple, intuitive proof that can, without too great a difficulty, be made rigorous: The teeth of the sawtooth graph of $f(x)$ have increasingly straight edges. Thus each tooth gives an area equal to about half the base. Thus the limit is 1/2.

6. A computer program intended to illustrate the workings of Newton's method has an unfortunate error. The program takes as input functions $f(x)$ for which $f(0) = 0$ and a starting value x_0, and it returns a sequence of numbers $x_1, x_2, \ldots$ that are supposed to get closer and closer to zero as Newton's method closes in on the (known) value $f(0) = 0$. But there is a mistake: The program uses the formula

$$x_{n+1} = 2\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)$$

where the factor of 2 is wrong. Nevertheless, the code sometimes outputs a sequence that converges to zero.

Prove that if $f(x) = x^2 + x$, then for any $x_0 > 0$,

(a) $x_{n+1} < x_n$ (This holds because if $x_n = x > 0$, then

$$x_{n+1} = 2\left(x - \frac{x^2 + x}{2x + 1}\right)$$

which simplifies to $2x^2/(2x+1)$. This is less than x because $2x^2 < x(2x+1) = 2x^2 + x$.

(b) The smaller x_n is, the smaller x_{n+1}/x_n is. If $x_n = x$, then the ratio x_{n+1}/x_n is $2x/(2x+1)$ as in the previous solution. Now this expression is increasing in x, (derivative is $2/(2x+1)^2 > 0$), so the larger x is, the larger the ratio, and more to the point, the smaller x is, the smaller this ratio is.

(c) $x_n \to 0$ as $n \to \infty$. If x_{n+1}/x_n keeps dropping, then starting with $x = x_0$, $x_n/x_0 < (x_1/x_0)^n$. Since $x_1/x_0 < 1$, the sequence (x_n) is sandwiched between 0 and a sequence that tends to zero.