Math 410.500: answers to exam 1

1. (a) \( f \) is analytic on \((a, b)\) iff for each \( x_0 \in (a, b) \) there exists a power series
\[
\sum_{k=0}^{\infty} a_k (x - x_0)^k
\]
and a number \( R > 0 \) so that \( f (x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \)
on \((x_0 - R, x_0 + R)\). (It’s too much to ask for that \( f \) equals a single
power series on all of \((a, b)\): \( \frac{1}{1+x^2} \) is analytic on \((-\infty, \infty)\), but no
single power series equals \( \frac{1}{1+x^2} \) for all \( x \).

(b) If there is a single number \( M \) so that \( \left| \sum_{k=1}^{n} f_k (x) \right| \leq M \) holds for all
\( n \in \mathbb{N} \) and all \( x \in E \), and if \( \{g_k (x)\} \) decreases monotonically and
uniformly to 0 on \( E \), then \( \sum_{k=1}^{\infty} f_k (x) g_k (x) \) converges uniformly on \( E \).

(c) For every \( \varepsilon > 0 \) there exists an \( N \) so that \( n, m \geq N \) implies that for
all \( x \in E \) there holds \( |f_n (x) - f_m (x)| < \varepsilon \).

(d) Either \( \sum k^k x^k \) or \( \sum k! x^k \) work. For both of these, the radius of
convergence is 0, so the power series converges only at \( x = 0 \).

2. We wish to show that \( f (x) \) is continuous everywhere in \( E \), so let \( x_0 \in E \)
be arbitrary. Take \( \varepsilon > 0 \). There exists \( N \) so that \( n \geq N \) implies that
for all \( x \in E \) we have \( |f_n (x) - f (x)| < \frac{\varepsilon}{3} \). Look specifically at \( f_N (x) \).
(This is important: there’s no reason to expect a \( \delta \) which will work for
all \( n \geq N \).) This is continuous on \( E \), hence at \( x_0 \), so there is a \( \delta > 0 \) so
that \( |x - x_0| < \delta \), \( x \in E \), implies that \( |f_N (x) - f_N (x_0)| < \frac{\varepsilon}{3} \). So, for any
\( x \in E \) with \( |x - x_0| < \delta \), we have
\[
|f (x) - f (x_0)| \leq |f (x) - f_N (x)| + |f_N (x) - f_N (x_0)| + |f_N (x_0) - f (x_0)| < \varepsilon,
\]
where the first and third terms are less that \( \frac{\varepsilon}{3} \) since \( f_N \) is uniformly within
\( \frac{\varepsilon}{3} \) of \( f \), and the second term is less than \( \frac{\varepsilon}{3} \) by how we found \( \delta \).

3. Pointwise convergence is simply saying that for each \( x \), the limit as \( n \) tends
to infinity exists. But \( \lim_{n \to \infty} \ln \left( x + \frac{1}{n} \right) = \ln x \), where we’re using the fact
that \( \ln \) is continuous on \((0, 1)\). It’s not hard to give a specific \( N, \varepsilon \) proof,
but that’s not what I was expecting. However, \( \left\{ \ln \left( x + \frac{1}{n} \right) \right\} \) does not
converge uniformly on \((0, 1)\). Here’s the proof: if the sequence converged
uniformly, the only possibility for a limit would be \( \ln (x) \), since uniform
convergence implies pointwise convergence. But \( \left| \ln \left( x + \frac{1}{n} \right) - \ln (x) \right| \)
can’t be made uniformly small on \((0, 1)\). For example, if you plug in \(\frac{1}{n}\) for \(x\), \(\left| \ln \left( \frac{1}{n} + \frac{1}{n} \right) - \ln \left( \frac{1}{n} \right) \right| = \ln (2)\). Thus no \(N\) exists so that \(n \geq N\) implies that \(\left| \ln \left( x + \frac{1}{n} \right) - \ln \left( \frac{1}{n} \right) \right| < \ln 2\) holds on \((0, 1)\), and the convergence is not uniform.

4. (a) Let’s try the M-test. On \((1, \infty)\) we have \(|ke^{-kx}| < ke^{-k}\), so we want to show that \(\sum_{k=1}^{\infty} ke^{-k}\) converges. By the root test: 

\[
\lim_{k \to \infty} \left| ke^{-k} \right|^{1/k} = \lim_{k \to \infty} k^{1/k} e^{-1} = \frac{1}{e} < 1,
\]

so \(\sum_{k=1}^{\infty} ke^{-k}\) converges (the ratio test works as well). Thus, by the M test, \(\sum_{k=1}^{\infty} ke^{-kx}\) converges uniformly on \((1, \infty)\).

(b) To use the theorem on term-by-term differentiation, we need to show that the series of derivatives converges uniformly, i.e., that \(\sum_{k=1}^{\infty} -k^2 e^{-kx}\) converges uniformly on \((1, \infty)\). Again using the M test, look at \(|-k^2 e^{-kx}| < k^2 e^{-k}\). The series \(\sum_{k=1}^{\infty} k^2 e^{-k}\) converges by the root (or ratio) test as in part a: 

\[
\lim_{k \to \infty} \left| k^2 e^{-k} \right|^{1/k} = \lim_{k \to \infty} k^{2/k} e^{-1} = \frac{1}{e} < 1.
\]

Hence by the M test, \(\sum_{k=1}^{\infty} -k^2 e^{-kx}\) converges uniformly on \((1, \infty)\).

We also have the original series converging at a point of \((1, \infty)\) (in fact, on all of it, by part a), so by term-by-term differentiation, \(f\) is differentiable, and 

\[
\frac{d}{dx} \left( \sum_{k=1}^{\infty} ke^{-kx} \right) = \sum_{k=1}^{\infty} \frac{d}{dx} (ke^{-kx}) = \sum_{k=1}^{\infty} -k^2 e^{-kx}.
\]

5. Let \(f(x) = (1 + 3x)^{1/2}\). By Taylor’s theorem, 

\[
f(x) = f(0) + xf'(0) + \frac{x^2}{2} f''(c),
\]

for some \(c\) between 0 and \(x\). Well, \(f'(x) = \frac{3}{2} (1 + 3x)^{-1/2}\) and \(f''(x) = -\frac{9}{4} (1 + 3x)^{-3/2}\), so we have 

\[
(1 + 3x)^{1/2} = 1 + \frac{3}{2} x - \frac{9}{8} (1 + 3c)^{-3/2} x^2,
\]

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for some $c$ between 0 and $x$. But, since $x$ is between 0 and 1, $c$ must also be between 0 and 1. Then $0 \leq c \leq 1$ implies $1 \leq 3c + 1 \leq 4$ implies

$$-\frac{9}{8} \leq -\frac{9}{8} (1 + 3c)^{-3/2} \leq -\frac{9}{64};$$

so

$$1 + \frac{3}{2} x - \frac{9}{8} x^2 \leq (1 + 3x)^{1/2} \leq 1 + \frac{3}{2} x - \frac{9}{64} x^2$$

for $x \in [0, 1]$. 