### Solutions to Exam I (501)

#### #1a) Factor and cancel

$$L := \lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{(x - 1)(x + 2)}{(x - 1)(x - 2)} = \lim_{x \to 1} \frac{x + 2}{x - 2} = -3$$

#### #1b) Factor creatively OR multiply by the conjugate (shown here)

$$L := \lim_{t \to 9} \frac{9 - t}{3 - \sqrt{t}} = \lim_{t \to 9} \frac{(9 - t)(3 + \sqrt{t})}{(3 - \sqrt{t})(3 + \sqrt{t})} = \lim_{t \to 9} \frac{9 - t}{3 + \sqrt{t}} = \lim_{t \to 9} 3 + \sqrt{t} = 6$$

#### #1c) nonzero/zero means infinite limit-determine the signs to find out which

$$L := \lim_{x \to (-2)} \frac{x - 1}{x^2 (x + 2)}$$

**numerator = -3, denominator = 0**

$x - 1$, negative, $x^2$, positive, $x + 2$, negative, $x < -2$

$$0 < \frac{x - 1}{x^2 (x + 2)}$$

**limit = $\infty$**

#### #1d) Divide by the dominating term ($x^3$)

$$L := \lim_{x \to \infty} \frac{x^2}{x^3 + 3x^2 - 10} = \lim_{x \to \infty} \frac{x^3 (x^3 + 3x^2 - 10)}{x^3 (x^3 + 3x^2 - 10)} = \lim_{x \to \infty} \frac{1}{x \left(1 + \frac{3}{x} - \frac{10}{x^3}\right)}$$

$$L := \lim_{x \to \infty} \frac{0}{1} = 0$$

#### #2 Use the power rule for the first term ($x^{-1/2}$) and the quotient rule for the second term
f(x) = \frac{1}{\sqrt{x}} + \frac{a x + b}{c x + d}

f(x) = x^2 + \frac{a x + b}{c x + d}

D(f)(x) = -\frac{1}{2} x + \frac{(c x + d) a - (a x + b) c}{(c x + d)^2}

#3 Subtract to find a vector (NOTE: must do end - start to get the right direction!), then divide by its magnitude to make it a unit vector:

\[ a := [-3, -2] - [5, 8] = [-8, -10] \]
\[ \text{mag} := \sqrt{(-8)^2 + (-10)^2} = \sqrt{164} \]
\[ u := \frac{[8, 10]}{\sqrt{164}} \]
\[ u := \left[ -\frac{8}{\sqrt{164}}, \frac{10}{\sqrt{164}} \right] \]

#4 Use \( \mathbf{r}(t) = \mathbf{r_0} + t \mathbf{v} \) The point given corresponds to \( \mathbf{r}_0 \) and the vector given is \( \mathbf{v} \)

\( \mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{v} \)
\( \mathbf{r}_0 = [1, 3] \)
\( \mathbf{v} = [-2, 5] \)

vector_eqn := \( \mathbf{r}(t) = [1, 3] + t [-2, 5] \)
parametric := \( [1 - 2 t, 3 + 5 t] \)

#5 Find the slope by computing \( f'(3) \) (using shortcut rules!) and find the point by computing \( f(3) \)

\( f(x) := x (1 - x) = x - x^2 \)
\( D(f)(x) = 1 - 2 x \)
\( \text{slope} := D(f)(3) = -5 \)
\( f(2) := 3 - 3^2 = -6 \)
\( \text{eqn} := y - (-6) = -5 (x - 3), y = -5 x + 9 \)

#6

\( f \) is not continuous at \( x = 1 \). \( \lim_{x \to 1^-} f(x) = 2, \lim_{x \to 1^+} f(x) = 4 \)

so \( \lim_{x \to 1} f(x) \) does not exist. But \( f(1) = 4 \), so the limit from the left does not equal the function value, but the limit from the right does. Therefore, \( f \) is continuous from the right, but not from the left.

#7 Use limit as \( h \to 0 \) (shown here) or limit as \( x \to 2 \)

\[ D(f)(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
\[ \frac{6}{1 + x + h} - \frac{6}{1 + x} \]
Get a common denominator on top and then multiply by \(1/h\):

\[
\lim_{h \to 0} \frac{1}{h} \left( \frac{6(1+x)}{(1+x+h)(1+x)} - \frac{6(1+x+h)}{(1+x+h)(1+x)} \right)
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{6+6x-6x-6h}{(1+x+h)(1+x)} \right)
\]

\[
= \lim_{h \to 0} \frac{1}{h} \left( \frac{1(-6h)}{h(1+x+h)(1+x)} \right)
\]

\[
= \lim_{h \to 0} \frac{6}{(1+x+h)(1+x)} = -\frac{6}{(1+x)^2}
\]

Therefore, the derivative at \(x = 2\) is 

\[-\frac{6}{(1+2)^2} = -\frac{2}{3}\]

\(\#8a\) \quad \lim_{x \to 4} f(x) = \infty \text{ if and only if, for any } N, \text{ there is a } \delta > 0 \text{ such that } f(x) > N \text{ whenever } |x - 4| < \delta

\(\#8b\) \quad \text{Choose a point } h \text{ units away from } x = a \text{ along the } x\text{-axis. The new point on the curve has coordinates } (a+h, f(a+h)). \text{ The slope of the secant line between this point and the original point } (a, f(a)) \text{ is } \frac{f(a+h) - f(a)}{a+h - a} \text{ or } \frac{f(a+h) - f(a)}{h}

\(\#9\) \quad \text{(written as 12 on the exam-sorry about that!)}

\text{a) } \lim_{x \to a} f(x) = L \text{ if and only if, for any } \epsilon > 0 \text{ there is a } \delta > 0 \text{ such that } |f(x) - L| < \epsilon \text{ whenever } |x - a| < \delta

\text{(Scratch work omitted here)} \quad \text{Let } f(x) = 6x + 1, a = 3, \text{ and } L = 19. \text{ Pick } \epsilon > 0 \text{ and let } \delta = \epsilon / 6.

\text{Then, if } |x - a| < \delta

\[
|x - 3| < \frac{\epsilon}{6}
\]

\[
6|x - 3| < \epsilon
\]

\[
6x - 18 < \epsilon
\]

\[
6x + 19 < \epsilon
\]

\[
|f(x) - L| < \epsilon
\]

Therefore, by definition, \(\lim_{x \to 3} 6x + 1 = 19\)

\text{b) Write the vectors in their component form:}

\text{Let } a = \langle a_1, a_2 \rangle, \ b = \langle b_1, b_2 \rangle, \text{ and } c = \langle c_1, c_2 \rangle

\text{Then } a \cdot (b + c) = \langle a_1, a_2 \rangle \cdot (\langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle)

= \langle a_1, a_2 \rangle \cdot \langle b_1 + c_1, b_2 + c_2 \rangle

= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2

\text{On the right-hand side, we have } (a \cdot b) + (a \cdot c) = (\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle) + (\langle a_1, a_2 \rangle \cdot \langle c_1, c_2 \rangle)

= (a_1b_1 + a_2b_2) + (a_1c_1 + a_2c_2)

= a_1b_1 + a_2b_2 + a_1c_1 + a_2c_2

\text{By commutativity, both sides of the equation are the same, therefore they must be equal, so}
\[ a \cdot (b + c) = (a \cdot b) + (a \cdot c) \]