Cohomological Consequences of the Pattern Map

Geometry and combinatorics on homogeneous spaces

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Lascoux and Schützenberger (1982) defined Schubert Polynomials, $S_w(x)$, for $w$ a permutation. These form a basis for all polynomials in $x$ and include all Schur polynomials.

For $w \in S_n$, they form a system of polynomial representatives of Schubert classes in the cohomology ring the flag variety $F\ell(n)$.

In "Structure de Hopf..." (‘82) L-S show that in the expansion

$$S_w(y_1, \ldots, y_n, z_1, \ldots, z_m) = \sum_{u,v} d_{w}^{u,v} \cdot S_u(y) \cdot S_v(z),$$

the coefficients $d_{w}^{u,v}$ are nonnegative.
Arbitrary Substitutions

Arbitrarily substituting $y$’s and $z$’s in some $G_w(x)$, e.g.,

$$G_w(y_1, y_2, z_1, y_3, y_4, z_2, z_3 \ldots) = \sum_{u,v} d_{w}^{u,v} G_u(y) G_v(z),$$

defines $d_{w}^{u,v} \in \mathbb{Z}$ depending on the positions of the $y$’s and $z$’s.

Bergeron–S. (‘98): These coefficients $d_{w}^{u,v}$ are naturally Schubert structure constants $c_{\beta,\gamma}^\alpha$ for multiplication in Schubert basis.

Used projection formula along the map $\mathbb{F}l(n) \times \mathbb{F}l(m) \rightarrow \mathbb{F}l(n+m)$ whose pullback gives the substitution.

Lenart, Robinson, S. (‘06): Generalized to the Grothendieck ring of $\mathbb{F}l(n+m)$. 

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Billey-Braden (‘03): \( G \): Semisimple linear algebraic group. Let \( \mathcal{F} \) be the flag variety of \( G \), parametrizing Borel subgroups. Let \( \eta \in G \) be semisimple. Set \( G_\eta := Z_G(\eta) \).

\[ B \mapsto B_\eta := B \cap G_\eta \] defines the geometric pattern map, \( \pi_\eta \),

\[ \mathcal{F}_\eta := \{ B \in \mathcal{F} \mid \eta \in B \} \overset{\pi_\eta}{\longrightarrow} G_\eta/B_\eta. \]

Let \( W, W_\eta \) be the Weyl groups of \( G, G_\eta \). If \( \pi_\eta : W \to W_\eta \) is the Billey-Postnikov generalised pattern map, then we have

Theorem [BB]. \( \pi_\eta(X_w \cap \mathcal{F}_\eta) = X_{\pi_\eta(w)}. \)

In type \( A \), the map \( \mathbb{F}\ell(n) \times \mathbb{F}\ell(m) \hookrightarrow \mathbb{F}\ell(n+m) \) is a section of \( \pi_\eta \), where \( \eta = \begin{pmatrix} \alpha I_n & 0 \\ 0 & \beta I_m \end{pmatrix} \).
Sections of the Pattern Map

Sections of the pattern map

\[ G_\eta/B_\eta \quad \xrightarrow{\text{III}} \quad \mathcal{F}_\eta \]

correspond to right cosets \( \text{III} \) of \( W_\eta \) in \( W \).

On cohomology rings, this is just a substitution of the canonical generators \( \mathfrak{h}^* \) of \( H^*(\mathcal{F}) \) for the canonical generators \( \mathfrak{h}^* (!) \) of \( H^*(G_\eta/B_\eta) \). (Here \( \eta \) lies in a maximal torus for \( G \), which is also a maximal torus for \( G_\eta \).)

In type A, this is just renaming and shuffling the variables in a Schubert polynomial. E.g.,

\[ \text{III}^*(\mathcal{S}_w) = \mathcal{S}_w(y_1, y_2, z_1, y_3, y_4, z_2, z_3, \ldots) . \]
Sections in Homology

For $u \in W_\eta$, $X_u B_\eta = X_u B_\eta \cap X_{\omega_\eta} B'_\eta$, where $B'_\eta$ is the Borel opposite to $B_\eta$ and $\omega_\eta$ is the longest element in $W_\eta$. Thus

$$\text{III}(X_u B_\eta) \subset X_{\text{III}(u)} B \cap X_{\text{III}'(\omega_\eta)} B'.$$

A dimension computation shows equality, so

$$\text{III}_*[X_u B_\eta] = [X_{\text{III}(u)} B \cap X_{\text{III}'(\omega_\eta)} B'] .$$

For $w \in W$, we have $\text{III}_*[S_w] = \sum_{u \in W_\eta} d^u_{w} S_u$, so $d^u_{w}$ is a particular Schubert structure constant.

$$p_*(\text{III}_*[S_w] \cap [X_u]) = p_*(S_w \cap [X_{\text{III}(u)} \cap X'_{\text{III}'(\omega_\eta)}]),$$

where $p$ is the map to a point.

$\leadsto d^u_{w}$ is a particular Schubert structure constant.
The Actual Formula

The section III corresponds to a right coset of $W_\eta$. Let $\varsigma \in W$ be the minimal length coset representative. Then $d^w_u = c^{u\varsigma}_{w,\varsigma}$.

Algorithm:

Expand the product $\mathcal{G}_w \cdot \mathcal{G}_\varsigma$ in Schubert basis for $H^*(\mathcal{F})$.
Restrict to terms of the form $\mathcal{G}_{u\varsigma}$ for $u \in W_\eta$.
Replace $\mathcal{G}_{u\varsigma}$ by $\mathcal{G}_u$ to obtain formula for $\text{III}^*(\mathcal{G}_w)$.

Example: $G = C_4$, $G_\eta = A_3$, and $\varsigma = 2 \ 1 \ 3 \ 4$

$\mathcal{C}_{3 \ 1 \ 4 \ 2} \cdot \mathcal{C}_{2 \ 1 \ 3 \ 4} = \mathcal{C}_{3 \ 2 \ 4 \ 1} + 2\mathcal{C}_{2 \ 3 \ 4 \ 1} + 2\mathcal{C}_{4 \ 3 \ 1 \ 2} + 2\mathcal{C}_{2 \ 3 \ 4 \ 1} + 2\mathcal{C}_{1 \ 4 \ 3 \ 2} + 2\mathcal{C}_{4 \ 2 \ 3 \ 1} \cdot$

so $\text{III}^*(\mathcal{C}_{3 \ 1 \ 4 \ 2}) = 2\mathcal{C}_{3412} + 2\mathcal{C}_{3241} + 2\mathcal{C}_{4132} + 2\mathcal{C}_{2431}$.