Khovanskii-Rolle Continuation for Real Solutions

Computational Algebraic and Analytic Geometry for Low-Dimensional Varieties

6 January 2009, AMS National Meeting

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Our numerical future

Increasing parallelism \(\Rightarrow\) The future of computation in algebraic geometry is numerical.

We want to find all real solutions to a system of equations.

Current dominant numerical algorithm for solving, homotopy continuation, necessarily computes all solutions, both real and complex.

Two classes of numerical algorithms for real solutions:

— Exclusion methods.
  Well-developed algorithms based on repeated subdivision.

— Semidefinite programming.
  Recently proposed by Lasserre, Laurent, and Rostalski.
A third method

Khovanskii-Rolle continuation is a third numerical method to compute real solutions.

— Based on proof of fewnomial bounds for real solutions.

— Uses 2 symbolic steps:

1) **Gale duality** reduces a (potentially high-degree) polynomial system to a system of rational functions on a different space.

2) Reducing this to solving some systems of low-degree polynomials & some path-continuation.

— Complexity is essentially the fewnomial bound.
Gale duality, via example

Suppose we have the system of polynomials,
\[
\begin{align*}
    v^2 w^3 &= 1 - u^2 v - uv^2 w, \\
    v^2 w &= \frac{1}{2} - u^2 v + uv^2 w, \\
    uvw^3 &= \frac{10}{11}(1 + u^2 v - 3uv^2 w).
\end{align*}
\] (1)

Observe that
\[
\begin{align*}
    (u^2 v)^2 \cdot (v^2 w^3)^3 &= (uv^2 w)^2 \cdot (v^2 w) \cdot (uvw^3)^2 \\
    (uv^2 w)^3 \cdot (v^2 w^3) &= (u^2 v) \cdot (v^2 w)^3 \cdot (uvw^3).
\end{align*}
\]

Substituting (1) into this, writing \(x\) for \(u^2 v\) and \(y\) for \(uv^2 w\), and solving for 0, gives the Gale system of master functions
\[
\begin{align*}
    f := x^2 (1-x-y)^3 - y^2 \left(\frac{1}{2} - x + y\right) \left(\frac{10}{11}(1+x-3y)\right)^2 &= 0, \\
    g := y^3 (1-x-y) - x \left(\frac{1}{2} - x + y\right)^3 \frac{10}{11}(1 + x - 3y) &= 0.
\end{align*}
\]
Gale duality, continued

The original system is equivalent to the Gale system

\[ f := x^2(1-x-y)^3 - y^2\left(\frac{1}{2} - x + y\right)\left(\frac{10}{11}(1+x-3y)\right)^2 = 0, \]
\[ g := y^3(1-x-y) - x\left(\frac{1}{2} - x + y\right)^3\frac{10}{11}(1 + x - 3y) = 0, \]

in the complement of the lines given by the linear factors.
Khovanskii-Rolle continuation

Given a system of master functions

\[ \prod_{i=1}^{\ell+n} p_i(x)^{a_{i,j}} = 1 \quad j = 1, \ldots, \ell, \quad (*) \]

\((p_i(x) \text{ linear})\), we find solutions in the polyhedron

\[ \Delta := \{ x \in \mathbb{R}^\ell \mid p_i(x) > 0 \} . \]

The Khovanskii-Rolle Theorem (next slide) reduces solving (*) to solving low degree polynomial systems, together with path continuation.

This is our new algorithm, which we now explain.
Khovanskii-Rolle Theorem

Theorem. *Between any two zeroes of* \( g \) *along the curve* \( V(f) : f = 0 \), *lies at least one zero of the Jacobian* \( df \wedge dg \).

Starting where \( V(f) \) meets the boundary of the polyhedron \( \Delta \) and where the Jacobian vanishes on \( V(f) \), tracing the curve \( V(f) \) in both directions finds all solutions \( f = g = 0 \).
Degree reduction \((\ell = 2)\)

A system of master functions

\[
\prod_{i=1}^{2+n} p_i(x)^{a_{i,j}} = 1 \quad j = 1, 2
\]

in logarithmic form

\[
\varphi_j := \sum_{i=1}^{2+n} a_{i,j} \log p_i(x) = 0 \quad j = 1, 2 ,
\]

has Jacobians of low degree

\[
J_2 := \text{Jac}(\varphi_1, \varphi_2) \quad J_1 := \text{Jac}(\varphi_1, J_2) .
\]

Here, \(n = \text{deg}(J_2)\) and \(2n = \text{deg}(J_1)\).
An example

Consider the system with $\ell = 2$ and $n = 4$:

\[
\begin{align*}
\mathbf{f}_1 & := \frac{(3500)^{12} x^{27} (3 - x)^8 (3 - y)^4}{y^{15} (4 - 2x + y)^{60} (2x - y + 1)^{60}} = 1, \\
\mathbf{f}_2 & := \frac{(3500)^{12} x^8 y^4 (3 - y)^{45}}{(3 - x)^{33} (4 - 2x + y)^{60} (2x - y + 1)^{60}} = 1.
\end{align*}
\]
Low-Degree Jacobians

If \( \varphi_i := \log(f_i) \), then \( J_2 := \text{Jac}(\varphi_1, \varphi_2) \cdot \prod p_i(x, y) = 
\begin{align*}
2736 - 15476x + 2564y + 32874x^2 - 21075xy + 6969y^2 - 10060x^3
- 7576x^2y + 8041xy^2 - 869y^3 + 7680x^3y - 7680x^2y^2 + 1920xy^3.
\end{align*}
(polynomial of degree \( n = 4 \).) \( J_1 := \text{Jac}(\varphi_1, \Gamma_2) \cdot \prod p_i(x, y)^2 = 
\begin{align*}
8357040x - 2492208y - 25754040x^2 + 4129596xy - 10847844y^2
- 37659600x^3 + 164344612x^2y - 65490898xy^2 + 17210718y^3 + 75054960x^4
- 249192492x^3y + 55060800x^2y^2 + 16767555xy^3 - 2952855y^4 - 36280440x^5
+ 143877620x^4y + 35420786x^3y^2 - 80032121x^2y^3 + 19035805xy^4 - 1128978y^5
+ 5432400x^6 - 33799848x^5y - 62600532x^4y^2 + 71422518x^3y^3 - 13347072x^2y^4
- 1836633xy^5 + 211167y^6 + 2358480x^6y + 21170832x^5y^2 - 13447848x^4y^3
- 8858976x^3y^4 + 7622421x^2y^5 - 1312365xy^6 - 1597440x^6y^2 - 1228800x^5y^3
+ 4239360x^4y^4 - 2519040x^3y^5 + 453120x^2y^6.
\end{align*}
(A polynomial of degree \( 8 = 2n \).)

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Completing the example

Follow $V(J_2) \cap \partial \Delta$ and $J_1 = J_2 = 0$ along $V(J_2)$ to find $J_2 = \varphi_1 = 0$.

Follow $V(\varphi_1) \cap \partial \Delta$ and $\varphi_1 = J_2 = 0$ along $V(\varphi_1)$ to find $\varphi_1 = \varphi_2 = 0$. 

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