WACHSPRESS VARIETIES

A Dissertation

by

COREY IRVING

Submitted to the Office of Graduate Studies of Texas A&M University in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

December 2012

Major Subject: Mathematics
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ABSTRACT

Wachspress Varieties. (December 2012)

Corey Irving, M.S., The University of Massachusetts;
B.S., The University of Massachusetts
Chair of Advisory Committee: Frank Sottile

Barycentric coordinates are functions on a polygon, one for each vertex, whose values are coefficients that provide an expression of a point of the polygon as a convex combination of the vertices. Wachspress barycentric coordinates are barycentric coordinates that are defined by rational functions of minimal degree. We study the rational map on $\mathbb{P}^2$ defined by Wachspress barycentric coordinates, the Wachspress map, and we describe polynomials that set-theoretically cut out the closure of the image, the Wachspress variety. The map has base points at the intersection points of non-adjacent edges.

The Wachspress map embeds the polygon into projective space of dimension one less than the number of vertices. Adjacent edges are mapped to lines meeting at the image of the vertex common to both edges, and blows up the base points into lines. The deformed image of the polygon is such that its non-adjacent edges no longer intersect but both meet the exceptional line over the blown-up corresponding base point.

We find an ideal that cuts out the Wachspress variety set-theoretically. The ideal is generated by quadratics and cubics with simple expressions along with other polynomials of higher degree. The quadratic generators are scalar products of vectors of linear forms and the cubics are determinants of $3 \times 3$ matrices of linear forms. Finally, we conjecture that the higher degree generators are not needed, thus the ideal is generated in degrees two and three.
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To My wife Lorena
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CHAPTER I

INTRODUCTION

A. Introduction

Introduced by Möbius in 1827 [1], barycentric coordinates for triangles are classical, but their generalization to arbitrary polygons is an area of current research. Barycentric coordinates are functions on a polygon, one for each vertex, that express a point in the polygon as a convex combination of the vertices. In other words, if \( v_1, \ldots, v_N \) are the vertices of an \( N \)-gon, a polygon with \( N \) sides, then barycentric coordinates are functions \( \beta_1, \ldots, \beta_N \) on the \( N \)-gon such that

\[
p = \sum_{i=1}^{N} \beta_i(p)v_i \quad \text{and} \quad \sum_{i=1}^{N} \beta_i(p) = 1,
\]

with the coefficients \( \beta_i(p) \) non-negative for each point \( p \) in the \( N \)-gon.

There is only one way to define barycentric coordinates for triangles; in this case each coordinate function \( \beta_i \) is a linear polynomial. However, they are not unique for larger polygons. For a square, there is one type of barycentric coordinates where each \( \beta_i \) is a piecewise linear function and another type where each \( \beta_i \) is a quadratic polynomial, see Example (I.8). Rarely can barycentric coordinates for non-triangles be expressed as functions as simple as polynomials. The next best that one could hope for are rational expressions. Rational barycentric coordinates exist for any convex polygon and they are our focus in this work.

While convenient to work with mathematically, these rational coordinates are often not the best choice for applications. Barycentric coordinates are used in geometric

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modeling to deform shapes. Rational coordinates sometimes produce deformations that are too crude; others with more complicated expressions often produce better quality deformations. For many applications there is current research investigating which barycentric coordinates have the best properties for those applications [2, 3]. Deforming planar shapes can be accomplished by placing the shape inside of a polygon, then the vertices are moved and barycentric coordinates are used to extend the motion to the entire shape [4]. This technique is employed frequently by computer animation studios [5]. Barycentric coordinates are fundamental in the construction of *multisided surface patches*, which are used in computer aided geometric design [6]. A patch is a deformed image of a planar polygon that has been transformed in a controlled way to have desired smoothness properties and to satisfy certain boundary conditions. Much of this theory was developed for and pioneered by the automobile industry [6].

In 1975 Eugene Wachspress introduced rational barycentric coordinates for polygons in his work on finite elements [7]. In 1996 Joe Warren showed that Wachspress’s coordinates are in some sense the simplest possible. He showed they are the unique rational barycentric coordinates of minimal degree [8, 9].

Wachspress coordinates define a rational map $\beta$, the *Wachspress map*, on the projective plane $\mathbb{P}^2$ whose value at a point $p$ of the polygon is the $N$-tuple of barycentric coordinates of $p$ considered as a point in $\mathbb{P}^{N-1}$. This map has $N(N - 3)/2$ base points occurring at the polygon’s *external vertices*, points where non-adjacent edges meet. It is a birational isomorphism onto its image whose inverse $\tau$ is projection from a codimension three plane $\mathcal{C}$, called the *center of projection*. The Zariski closure of $\beta$, the *Wachspress variety* $\mathcal{W}$, is our main object of study. It is a surface in $\mathbb{P}^{N-1}$ and our goal is to describe an ideal that cuts out $\mathcal{W}$ set-theoretically.

In Chapter 2, we use linear algebra to compute the ideal of $\mathcal{W}$ in degree two.
These quadratics cut out the union $\mathcal{W} \cup \mathcal{C}$. In Chapter 3, cubics in the ideal are described that cut out the Wachspress variety in $\mathcal{C}$. Finally, we show that the quadratics from Chapter 2, the cubics from Chapter 3, and some additional higher dimensional polynomials cut out the Wachspress variety set-theoretically, and later conjecture that the higher degree polynomials are not needed so the Wachspress variety is cut out in degrees two and three. The Wachspress quadrics and cubics admit an elegant description. Each quadratic is expressed as a scalar product of two vectors of linear forms while each cubic is the determinant of a $3 \times 3$ matrix of linear forms. Later in Chapter 3 we describe syzygies among the Wachspress quadratics and cubics and work out the free resolution of the Wachspress variety in some special cases. Lastly, we look at examples of Wachspress varieties and the ideal generated by the Wachspress quadratics and cubics for some small values of $N$.

For a quadrilateral, $\mathcal{W}$ is a quadric hypersurface in $\mathbb{P}^3$ and the center of projection is a point on $\mathcal{W}$. For the pentagon, $\mathcal{W}$ is the intersection of two quadric hypersurfaces and $\mathcal{C}$ is a line. The Wachspress variety was examined in [10] for a particular hexagon. There it was observed that in this case $\mathcal{W}$ is cut out by three quadratics and one cubic in $\mathbb{P}^5$. The variety defined by the quadratics is $\mathcal{W} \cup \mathcal{C}$, and, in $\mathcal{C}$, the cubic cuts out the intersection curve $\mathcal{W} \cap \mathcal{C}$. Also, it was noted that the intersection curve $\mathcal{W} \cap \mathcal{C}$ is a reducible cubic in $\mathcal{C}$. We will see that the this curve is reducible because of the symmetries of this hexagon.

The Wachspress variety has interesting geometry. The image of vertex $v_i$ under $\beta$ is $\hat{v}_i := [0 : \cdots : 1 : \cdots : 0]$ where the 1 is in the $i$th position, and the image of the edge through $v_i$ and $v_{i+1}$ is the line through $\hat{v}_i$ and $\hat{v}_{i+1}$ in $\mathcal{W} \subseteq \mathbb{P}^{N-1}$. The Wachpress map deforms the $N$-gon in such a way that non-adjacent edges no longer meet, although they do meet a common line which is the exceptional divisor over the corresponding external vertex.
In $\mathbb{P}^2$ there is a unique degree $N - 3$ curve, the adjoint curve $\mathcal{A}$, passing through the $N(N - 3)/2$ external vertices. The Wachspress map takes the adjoint curve obtained by intersecting $\mathcal{W}$ with $\mathcal{C}$. The proper transform of the adjoint curve under $\beta$ is $\mathcal{W} \cap \mathcal{C}$. The $\tau$-fiber over a generic point of $\mathcal{A}$ is a point on $\mathcal{C}$ while over an external vertex it is a line passing through $\mathcal{C}$.

B. Background

1. Algebraic Geometry

We quickly review some fundamental ideas from algebraic geometry. The presentation here is based on Harris [11]. Complex affine $n$-space, $\mathbb{A}^n$, is the set of $n$-tuples of complex numbers. An affine variety in $\mathbb{A}^n$ is the zero set of a collection of polynomials in $n$ variables. Complex projective $n$-space $\mathbb{P}^n$ is the space of lines in $\mathbb{A}^{n+1}_{\mathbb{C}}$ through the origin. More precisely, it is $\mathbb{A}^{n+1}_{\mathbb{C}} \setminus \{0\}$ with points $x$ and $y$ identified if there is a nonzero complex number $\lambda$ with $x = \lambda y$. A point in $\mathbb{P}^n$ will be denoted by $[x_0 : \cdots : x_n]$. A polynomial is homogeneous if all its terms have the same degree. A projective variety in $\mathbb{P}^n$ is the zero set of a collection of homogeneous polynomials in $n + 1$ variables. A variety is irreducible if it can not be written as the union of two of its proper subvarieties. Every variety has a unique decomposition into irreducible subvarieties. The subvarieties in this decomposition are the components of the variety.

Let $R = \mathbb{C}[x_0, \ldots, x_n]$ be the ring of polynomials in $n + 1$ variables with coefficients in $\mathbb{C}$. This ring is graded; it is the direct sum

$$R = \bigoplus_{d=0}^{\infty} R_d,$$

where $R_d$ is the vector space of homogeneous polynomials in $R$ of degree $d$ along with the zero polynomial. An ideal in $R$ is homogeneous if it can be generated by
homogeneous polynomials. For an ideal $I \subseteq R$ we set $I_k := I \cap R_d$, the degree $k$ piece of $I$. We will assume that all ideals are homogeneous.

A homogeneous ideal $I$ defines a projective variety $\mathcal{V}(I)$, the common zero locus of all polynomials in $I$. Given a variety $X \subseteq \mathbb{P}^n$, the ideal of all polynomials vanishing on $X$ is denoted $I(X)$. If $I \subseteq R$ is an ideal and $X = \mathcal{V}(I)$ we say that $I$ cuts out $X$. An ideal $I$ may cut out a variety $X$ but not contain all polynomials vanishing on $X$. Primarily because of this, we consider three different refinements of the notion of an ideal cutting out a variety. If $I = I(X)$ we say $I$ cuts out $X$ ideal-theoretically. The saturation of $I$ is the ideal $\text{sat}(I) = \{ f \in R \mid (x_0, \ldots, x_n)^k I \subseteq I \text{ for some } k \}$. It arises because the ideal $(x_0, \ldots, x_n)$ defines the empty set in $\mathbb{P}^n$. Observe that we have $I \subseteq \text{sat}(I) \subseteq I(\mathcal{V}(I))$. If $\text{sat}(I) = I(X)$, then we say $I$ cuts out $X$ scheme-theoretically. Finally, if $\mathcal{V}(I) = X$ we say $I$ cuts out $X$ set-theoretically. Among these, we have the relations:

$$\text{ideal-theoretic} \Rightarrow \text{scheme-theoretic} \Rightarrow \text{set-theoretic}.$$  

We illustrate this distinction:

**Example I.1.** Set $X = \{[0 : 0 : 1]\}$. This is a variety in $\mathbb{P}^2$, for example, it is cut out ideal-, scheme-, and set-theoretically by its ideal $I(X) = (x_0, x_1)$. The ideal $(x_0, x_1, x_2) \cdot I(X) = (x_0^2, x_1^2, x_0x_1, x_0x_2, x_1x_2)$ cuts out $X$ set- and scheme-theoretically but not ideal-theoretically. The ideal $(x_0^2, x_1^2)$ only cuts out $X$ set-theoretically.

a. Birational Geometry

Let $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$ be projective varieties. A collection of $n$-variate polynomials $\phi_0, \ldots, \phi_m$ of the same degree such that $\mathcal{V}(\phi_0, \ldots, \phi_m) \cap X = \emptyset$ defines a morphism $\phi : X \to Y$ given by $x \mapsto [\phi_0(x) : \cdots : \phi_n(x)]$. Two such morphisms are equivalent if
their values agree on an open subset of \( X \). If \( X \not\subseteq \operatorname{V}(\phi_0, \ldots, \phi_m) \) the map \( \phi \) defines a \textit{rational map} \( \phi : X \to Y \). Morphisms are special cases of rational maps. A rational map does not necessarily have a value at all points of \( X \). Two rational maps are equivalent if they agree on an open subset of \( X \).

For a representation of the rational map \( \phi \) the image \( \phi(X) \) may not be a projective variety; however, it is contained in its \textit{Zariski closure}, the smallest projective variety containing \( \phi(X) \) or, equivalently, the intersection of all projective varieties containing \( \phi(X) \).

A birational isomorphism is a rational map \( \phi : X \to Y \) such that there exists another rational map \( \psi : Y \to X \) with both compositions \( \phi \circ \psi \) and \( \psi \circ \phi \) equivalent to the identity map on \( X \) and \( Y \), respectively.

The product \( X \times Y \) is a projective variety whose subvarieties are zero sets of collections of bihomogeneous polynomials. The graph of a rational map \( \phi : X \to Y \) denoted \( \Gamma_\phi \) is the closure in \( X \times Y \) of the subset

\[
\{(x, y) \mid y = \phi(x) \text{ on some open set where the map is defined}\}.
\]

If \( \phi \) is defined by polynomials \( \phi_0, \ldots, \phi_m \) and the saturation of the ideal generated by these polynomials is the ideal of the variety \( Z := \operatorname{V}(\phi_0, \ldots, \phi_m) \), then \( \Gamma_\phi \) is called the \textit{blowup of \( X \) along \( Z \)}. Let \( \pi : \Gamma_\phi \to X \) be projection onto the first coordinate. The fiber \( \pi^{-1}(Z) \) is the \textit{exceptional divisor} of the blowup.

It follows from Hironaka’s Theorem [12] that any rational map \( \phi : X \to \mathbb{P}^m \) may be \textit{resolved} by a sequence of blowups. More precisely:

\textbf{Theorem I.2.} \textit{Given a rational map} \( \phi : X \to \mathbb{P}^m \) \textit{there is a sequence of varieties} \( X = X_1, \ldots, X_k \), \textit{subvarieties} \( Z_i \subseteq X_i \) \textit{with} \( X_{i+1} \) \textit{a blowup of} \( X_i \) \textit{along} \( Z_i \), \textit{and projection maps} \( \pi_i : X_{i+1} \to X_i \) \textit{such that the composition} \( \pi_k \circ \pi_{k-1} \circ \cdots \pi_1 \circ \phi : X_k \to \mathbb{P}^m \).
Example I.3. Consider the rational map \( \phi : \mathbb{P}^2 \to \mathbb{P}^1 \) defined by \([x_0 : x_1 : x_2] \mapsto [x_0, x_1] \). Since the ideal \((x_0, x_1)\) is the ideal of the single point \([0 : 0 : 1]\) \(\in\mathbb{P}^2\), the graph of \(\phi\) is the blowup of \(\mathbb{P}^2\) along the variety \(\mathbb{V}(x_0, x_1) = \{[0 : 0 : 1]\}\). The map \(\phi\) is undefined at \([0 : 0 : 1]\), but according to Theorem I.2 we can resolve \(\phi\). Let \(\Gamma_\phi = \mathbb{V}(x_0 y_1 - x_1 y_0) \subseteq \mathbb{P}^2 \times \mathbb{P}^1\) be the graph of \(\phi\) and \(\pi_2 : \mathbb{P}^2 \to \mathbb{P}^1\) the projection onto the second coordinate. We claim \(\pi_2\) resolves \(\phi\). Let \(\pi_1 : \Gamma_\phi \to \mathbb{P}^2\) be the projection to the first coordinate, the blowup map. Let \(P := ([x_0 : x_1 : x_2], [y_0 : y_1]) \in \Gamma_\phi\). Then \(\phi \circ \pi_1(P) = \phi([x_0 : x_1 : x_2]) = [x_0 : x_1]\) and \(\pi_2(P) = [y_0 : y_1]\). Note that these two values to agree by the defining equation of the graph.

Theorem I.2 simplifies greatly if \(X\) is a surface and this is the case for the applications of the theorem used in this work, in fact \(X\) will be \(\mathbb{P}^2\). The subvarieties \(Z_i\) being blown up are just points and the exceptional divisors are copies of \(\mathbb{P}^1\).

2. Barycentric Coordinates

Let \(\Delta\) be an \(N\)-gon with vertices \(v_1, \ldots, v_N\) and indices taken modulo \(N\) so that, for example, \(v_{N+1} = v_1\).

Definition I.4. Functions \(\{\beta_i : \Delta \to \mathbb{R} \mid 1 \leq i \leq N\}\) are barycentric coordinates if

1. \(\beta_i(p) \geq 0\)
2. \(p = \sum_{i=1}^{N} \beta_i(p)v_i\)
3. \(\sum_{i=1}^{N} \beta_i(p) = 1\)

for all \(p \in \Delta\).
Example I.5. (Line Segment)

Let $\Delta$ be the line segment between two vertices $v_1$ and $v_2$ in $\mathbb{R}^2$. The functions

$$
\beta_1(p) = \frac{d(v_2, p)}{d(v_1, v_2)}, \quad \beta_2(p) = \frac{d(p, v_1)}{d(v_1, v_2)},
$$

where $d$ is the Euclidean distance function, are barycentric coordinates.

Example I.6. (Triangle)

Barycentric coordinates of triangles can be described in terms of areas of subtriangles shown in Figure 1. The barycentric coordinate for the $i$-th vertex is $\beta_i = A_i/(A_0 + A_1 + A_2)$ where $A_i$ is the area indicated in the figure.

Fig. 1. Barycentric coordinates for a triangle

Example I.7 (Simplex). There is a similar description of the barycentric coordinates of simplices by splitting up into subsimplices.

Barycentric coordinates for simplices are unique; i.e., there is only one collection of functions $\beta_1, \ldots, \beta_N$ satisfying Definition I.4. To see this, fix a point $p$ of an $(N-1)$-simplex in $\mathbb{R}^{N-1}$. The unknown coefficients $\beta_1(p), \ldots, \beta_N(p)$ have $N$ independent conditions imposed on them by (2) and (3), hence are uniquely determined. However, barycentric coordinates for general polygons are not unique, many different types have been studied [2, 13, 8]. To illustrate non-uniqueness, we next describe two different ways to define barycentric coordinates for a square.
Example I.8. (Square)

We describe two sets of barycentric coordinates for the square in Figure 2. Here is a piecewise linear set:

\[ v_4 = (0, 1) \quad v_3 = (1, 1) \]
\[ v_1 = (0, 0) \quad v_2 = (1, 0) \]

Fig. 2. A square

\[ \beta_1 = \begin{cases} 
1 - x, & \text{if } x \leq y, \\
1 - y, & \text{if } x > y. 
\end{cases} \]
\[ \beta_2 = \begin{cases} 
x - y, & \text{if } x \leq y, \\
0, & \text{if } x > y. 
\end{cases} \]
\[ \beta_3 = \begin{cases} 
y, & \text{if } x \leq y, \\
x, & \text{if } x > y. 
\end{cases} \]
\[ \beta_4 = \begin{cases} 
0, & \text{if } x \leq y, \\
y - x, & \text{if } x > y. 
\end{cases} \]

and here is a set given by quadratic polynomials:

\[ \beta_1 = (1 - x)(1 - y) \quad \beta_2 = y(1 - x) \]
\[ \beta_3 = xy \quad \beta_4 = x(1 - y). \]

The second set of barycentric coordinates in Example I.8 are *Wachspress barycentric coordinates*, which were developed by Eugene Wachspress in 1975 [7] for an application to approximation theory. In this case they are a bidegree (1,1) tensor product Bézier patch [14]. Warren generalized them to higher dimensional polytopes in 1996 [8]. These coordinates are algebraic; each coordinate \( \beta_i \) is a rational function. They
are the unique rational functions satisfying (1)-(3) in Definition I.4 of minimal degree [9].

Wachspress coordinates admit a geometric description similar to that of barycentric coordinates for the triangle in terms of areas of subtriangles. Let \( A(a, b, c) \) denote the area of the triangle with vertices \( a, b, \) and \( c \).

**Definition I.9.** For \( 1 \leq j \leq N \) set \( \alpha_j := A(v_{j-1}, v_j, v_{j+1}) \) and \( A_j := A(p, v_j, v_{j+1}) \). The functions

\[
\beta_i = \frac{b_i}{\sum_{j=1}^{N} b_j},
\]

where \( b_i = \alpha_i \prod_{j \neq i - 1, i} A_j \) are Wachspress barycentric coordinates, see Figure 3.

![Fig. 3. Wachspress coordinates for a polygon](image)

**Remark I.10.** To simplify our expressions and take advantage of multilinear algebra we identify each vertex \( v_i \) with its lift \((1, v_i)\) at height 1 in three dimensional space. The vertices now span a cone through the origin with edge \([v_i, v_{i+1}]\) spanning a facet with normal vector \( n_i := v_i \times v_{i+1} \). We redefine \( \alpha_j \) and \( A_j \) to be the determinants \( \det(v_{j-1}, v_j, v_{j+1}) \) and \( \det(v_j, v_{j+1}, p) \), respectively, which agrees with the previous definitions and allows us to define Wachspress coordinates for non-convex polygons.

The non-negativity property of barycentric coordinates fails when \( \Delta \) is non-convex, but this is not a problem for us. Each \( A_j \) is a linear polynomial in \( p \); in particular,
set \( p := (1, x, y) \), then \( \ell_j := A_j = n_j \cdot p = \det(v_j, v_{j+1}, p) \). The linear polynomial \( \ell_j \) cuts out the line supporting the edge \([v_j, v_{j+1}]\). We see that the numerator \( b_i \) of each Wachspress coordinate is the product of \( N - 2 \) linear forms. With this algebraic definition we can allow complex vertices. It is important to note that all algebraic results we describe in this work hold in this generality, but most applications will take the convex \( \mathbb{R}^2 \) case. Our results however do require the condition that no three edges are concurrent, which is equivalent to the condition that \( |n_i \ n_j \ n_k| := \det(n_i, n_j, n_k) \neq 0 \) for all distinct indices \( 1 \leq i, j, k \leq N \).

**Definition I.11.** The **dual cone** to \( \Delta \) is the cone spanned by the normals \( n_1, \ldots, n_N \) and is denoted \( \Delta^* \).

### 3. Wachspress Varieties

We homogenize the numerators of Wachspress coordinates with a new variable \( z \) and let \( \mathbb{P}^\Delta \) be the projective space with coordinates indexed by the vertices of the polygon \( \Delta \).

**Definition I.12.** The **Wachspress map** is the rational map \( \beta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^\Delta \) sending \([z, x, y]\) to \([b_1(z, x, y), \ldots, b_N(z, x, y)]\).

This map assigns to each point of \( \Delta \) its Wachspress coordinates. The **Zariski closure** of a subset \( S \) of \( \mathbb{P}^\Delta \) is the smallest projective variety containing \( S \).

**Definition I.13.** The **Wachspress variety** \( \mathcal{W} \) is the Zariski closure of the image of the map \( \beta \).

**Lemma I.14.** The map \( \beta \) has \( N(N - 3)/2 \) base points occurring at the external vertices of \( \Delta \), see Figure 4.
Proof. By definition of Wachspress coordinates, the external vertices are basepoints as every Wachspress coordinate vanishes at them. Now assume that $p$ is an arbitrary basepoint of $\beta$. Since $b_1(p) = 0$ there exists an $i_1 \neq 1, N$ such that $\ell_{i_1} = 0$. We also have $b_{i_1}(p) = 0$, which means that there is some $i_2 \neq i_1 - 1, i_1$ such that $\ell_{i_2}(p) = 0$. Since $i_2 \neq i_1$ we can conclude that for any base point $p$ at least two of the lines $\ell_1, \ldots, \ell_N$ vanish at $p$. For an arbitrary basepoint $p$, we know that two lines $\ell_i$ and $\ell_j$ vanish at $p$. Suppose these lines are adjacent; say without loss of generality that $j = i - 1$. Then $p$ lies on the adjacent lines $\ell_{i-1}$ and $\ell_i$. But this means that $p$ is the vertex $v_i$, and this is not a basepoint since $b_i(v_i) \neq 0$. The non-adjacent edges of $\Delta$ are in one-to-one correspondence with the diagonals of the dual cone $\Delta^*$, thus there are $N(N - 3)/2$ base points. \hfill \Box

Theorem I.15. The degree of the Wachspress variety of an $N$-gon is

$$\frac{1}{2}(N^2 - 5N + 8).$$

Proof. We will use the following result known as the Degree Formula: If the dimension of the image of a rational map between projective spaces is two, the map is degree 1 and defined by degree $d$ polynomials, then the degree of the image is
d^2 − \{the number of base points counted with multiplicity\} [15]. The degree of the Wachspress map is 1, is defined by \((N − 2)\) polynomials, and the number of base points is \(N(N − 3)/2\).

We show that the base points have multiplicity one. Let \(p_{ij} := \ell_i \cap \ell_j\) be a base point. We can choose affine coordinates for \(p^2\) so that neither \(\ell_i\) nor \(\ell_j\) is the line at infinity. Then these lines form a normal crossing and we can choose our coordinates \((x, y)\) such that \(\ell_i = x, \ell_j = y,\) and \(p = (0, 0)\). In these coordinates we have

\[
b_i = \alpha_i \prod_{m \neq i, i} \ell_m, \quad \text{and} \quad \frac{\partial b_i}{\partial y}(p) = \ell_{i+1}(p) \cdots \ell_{j-1}(p)\ell_{j+1}(p) \cdots \ell_{i-1}(p).
\]

If this partial derivative were zero, then \(p\) would lie on at least three edges which is only possible if three edges are concurrent and this is not the case by our assumptions on \(\Delta\). Finally, we observe that \((N − 2)^2 − N(N − 3)/2 = \frac{1}{2}(N^2 − 5N + 8). \)

C. The Wachspress variety as a Blow-up of \(\mathbb{P}^2\)

By Theorem I.2 in the case of surfaces there is a blow-up \(\mathbb{P}^2\) at a finite number of points that resolves \(\beta\). A set of points that will accomplish this for us is the set \(Y\) of the \(N(N − 3)/2\) external vertices of \(\Delta\), which are the base points of \(\beta\).
Since $\beta$ is a birational isomorphism onto its image, resolving $\beta$ by blowing up $\mathbb{P}^2$ along $Y$ yields an isomorphism $\tilde{\beta} : \text{Bl}_Y(\mathbb{P}^2) \to \mathcal{W}$ taking an exceptional line $\pi^{-1}(p)$ over a base point $p$ to the line $\tau^{-1}(p)$ in $\mathcal{W}$.

D. Adjoint Polynomials

A polygon $\Delta$ defines a polyhedral cone in three space by putting the polygon in a plane at height one and taking all rays that pass through the origin and a point on an edge of $\Delta$. A triangulation of the polygon $\Delta$ will give a triangulation of the corresponding cone into simplices. Each triangle in a triangulation of polygon corresponds to a three-simplex $S$ spanned by vertices $v_i, v_j,$ and $v_k,$ whose volume is $a_S := |v_i v_j v_k| := \det(v_i, v_j, v_k)$.

**Definition I.16.** Let $C$ be a cone defined by a polygon, $v(C)$ its vertex set, and $T(C)$ a triangulation of $C$. The adjoint of $C$ is defined by

$$\mathcal{A}(z) := \sum_{S \in T(C)} a_S \prod_{v \in v(C) \setminus v(S)} (v \cdot z).$$

The adjoint is a tri-variate homogeneous polynomial of degree $N - 3$.

**Example I.17.** We calculate the adjoint polynomial of a quadrilateral using the triangulation in Figure 5. The adjoint in this case is

$$\mathcal{A}(z) = |v_1 v_2 v_4| v_3 \cdot x + |v_2 v_3 v_4| v_1 \cdot z,$$

Fig. 5. A triangulation of a quadrilateral
where we take \( z = (z_1, z_2, z_3) \) to be coordinates on \( \mathbb{P}^2 \).

**Theorem I.18.** *(Warren [8])*  
Adjoints are independent of the triangulation used.

The following will be used in several proofs throughout this work.

**Lemma I.19.** Any \( v_1, v_2, v_3, \) and \( v_4 \) vectors in three space satisfy

\[
\begin{vmatrix} v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 \\ v_3 & v_4 & v_1 \end{vmatrix} = 0.
\]

![Fig. 6. Triangulations of Quadrilateral](image)

**Proof.** Theorem I.18 applied to the adjoints computed using the two triangulations of the quadrilateral in Figure 6 yields

\[
\begin{vmatrix} v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 \\ v_3 & v_4 & v_1 \end{vmatrix} = 0.
\]

for and \( z \in \mathbb{P}^2 \). Subtracting and taking out the \( z \) factor produces

\[
\left( \begin{vmatrix} v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 \\ v_3 & v_4 & v_1 \end{vmatrix} - \begin{vmatrix} v_2 & v_3 & v_4 \\ v_1 & v_2 & v_3 \\ v_3 & v_4 & v_1 \end{vmatrix} \right) \cdot z = 0.
\]

This is zero for all \( z \) only if the factor in parentheses is zero, proving the result. Its also not hard to show directly that the result holds if the vectors are not affinely independent. \( \Box \)

**Theorem I.20.** *(Wachspress [7], Warren [8])*  
Wachspress coordinates reduce to linear interpolation on the edges of \( \Delta \).
This means that any point, \( p \), on the edge \([v_i, v_{i+1}]\) of \( \Delta \) is written \( p = \beta_i(p)v_i + \beta_{i+1}v_{i+1} \) where \( \beta_{i+1}(p) = \frac{|p - v_i|}{|v_{i+1} - v_i|} \) and \( \beta_i = \frac{|v_{i+1} - p|}{|v_{i+1} - v_i|} \). Here we take note of an important fact: the Wachspress coordinates of the vertex \( v_i \) are \((0, \ldots, 0, \beta_i(v_i) = 1, 0, \ldots, 0)\).

**Lemma I.21.** There are no linear algebraic relations among Wachspress coordinates.

*Proof.* Suppose \( \sum_{i=1}^N c_i \beta_i = 0 \) for some \( c_i \in \mathbb{C} \). For each \( i \) all Wachpress coordinates vanish at \( v_i \) except \( \beta_i \). The dependence relation reduces to \( c_i \beta(v_i) = 0 \) at \( v_i \). Since \( \beta_i(v_i) \neq 0 \), we have \( c_i = 0 \). Therefore there are no linear relations among Wachspress coordinates. \( \square \)

E. Image of Adjoint Curve Contained in Center

We conclude this Chapter by looking at what happens to the adjoint curve in \( \mathbb{P}^2 \) under the Wachspress map \( \beta \). In Warren’s work it is noted that the denominator of each Wachspress coordinate is the adjoint of the dual polygon \( \mathcal{A}(\Delta^*) \). This follows since both \( \sum_{i=1}^N b_i \) and \( \mathcal{A}(\Delta^*) \) are degree \( N - 3 \) polynomials interpolating the \( N(N-3)/2 \) base points of \( \beta \). From this observation we can show the following.

**Lemma I.22.** The image of adjoint curve under the Wachspress mapping \( \beta \) is contained in center of projection.

*Proof.* The adjoint curve \( A \) is the curve in \( \mathbb{P}^2 \) cut out by \( \sum_{S \in T(\Delta^*)} a_S \prod_{\ell \in S} \ell_j \) for any triangulation \( T(\Delta^*) \) of \( \Delta^* \). For any point \( z := [z_0 : z_1 : z_2] \in \mathbb{P}^2 \) we have

\[
\sum_{i=1}^N b_i(z)v_i = \mathcal{A}(\Delta^*)(z) = z
\]
where defined by definition of barycentric coordinates. Thus the equation

$$\sum_{i=1}^{N} b_i(z)v_i = \begin{pmatrix}
    z \mathcal{A}(\Delta^*)(z) \\
    x \mathcal{A}(\Delta^*)(z) \\
    y \mathcal{A}(\Delta^*)(z)
\end{pmatrix}$$

must hold for all $z$. The right hand side vanishes because $z$ is on the adjoint curve. Thus $0 = \sum_{i=1}^{N} b_i(z)v_i = \pi(\beta(z))$ and hence $\beta(x) \in \mathcal{C}$ as desired.

Since the center of projection $\mathcal{C}$ has codimension three and the Wachspress variety $\mathcal{W}$ is two-dimensional, we expect that the intersection $\mathcal{C} \cap \mathcal{W}$ is a curve on $\mathcal{W}$. Since the image $\beta(\mathcal{A}(\Delta^*))$ of the adjoint curve is clearly contained in $\mathcal{W}$ and by Lemma I.22 contained in $\mathcal{W}$ we conclude that $\mathcal{C} \cap \mathcal{W} = \beta(\mathcal{A}(\Delta^*))$.
CHAPTER II

WACHSPRESS QUADRATICS

A. Introduction

We describe the quadratic polynomials that vanish on the Wachspress variety $\mathcal{W}$, the Wachspress quadratics, and study the geometry of the variety they cut out. We characterize these quadratics by showing they must vanish on a certain linear space and finding a set of monomials that support them. To understand the geometry of the variety they define, we will describe the variety’s irreducible decomposition.

The set of polynomials vanishing on the variety $\mathcal{W}$ is the Wachspress ideal $I$. We construct a generating set for $I_2$ consisting of polynomials that are each expressed as a scalar product with a fixed vector $\tau$. This vector $\tau$ defines a rational map $\mathbb{P}^\Delta \dashrightarrow \mathbb{P}^2$, also denoted by $\tau$, given by

$$
\mathbf{x} \mapsto \sum_{i=1}^{N} x_i v_i,
$$

where $\mathbf{x} = [x_1 : \cdots : x_N] \in \mathbb{P}^\Delta$, called the linear projection. By Property 2 of the definition of barycentric coordinates, the composition $\tau \circ \beta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a birational isomorphism and hence $\dim(\mathcal{W}) = 2$. Since $v_i \in \mathbb{C}^3$, the vector $\tau$ is a triple of linear forms in $\mathbb{C}[\mathbb{P}^\Delta]$. The linear subspace $\mathcal{C}$ of $\mathbb{P}^\Delta$ where the linear projection is undefined is called the center of projection. The ideal $I(\mathcal{C})$ of $\mathcal{C}$ is generated by the three linear forms defining $\tau$.

Lastly, we show that $\mathcal{V}(\langle I_2 \rangle) = \mathcal{W} \cup \mathcal{C}$ is an irreducible decomposition. This will be useful in Chapter 3, where polynomials are described that cut out $\mathcal{W}$. These polynomials consist of a generating set for $I_2$ and cubics that cut out $\mathcal{W}$ in $\mathcal{C}$. 

B. Diagonal Monomials

Polynomials in $I_2$ are supported on a special set of quadratic monomials. A \textit{diagonal monomial} is a monomial $x_ix_j$ in $\mathbb{C}[\mathbb{P}^\Delta]$ such that $j \notin \{i - 1, i, i + 1\}$. Identifying variable $x_i$ with vertex $v_i$, a diagonal monomial is a diagonal in $\Delta$, see Figure 7.

\textbf{Lemma II.1.} \textit{Polynomials in $I_2$ are linear combinations of diagonal monomials.}

\textit{Proof.} Let $Q$ be a polynomial in $I_2$. Then $Q(\beta) = Q(b_1, \ldots, b_N) = 0$. On the edge $[v_k, v_{k+1}]$ all the $b_i$ vanish except $b_k$ and $b_{k+1}$. Thus on this edge, the expression $Q(\beta) = 0$ is

$$c_1b_k^2 + c_2b_kb_{k+1} + c_3b_{k+1}^2 = 0 \quad (2.1)$$

for some constants $c_1$, $c_2$, and $c_3$ in $\mathbb{C}$. Recall that $b_i(v_j) = 0$ if $i \neq j$ and $b_i(v_i) \neq 0$ for each $i$. Evaluating Equation 2.1 at $v_k$ and $v_{k+1}$, we conclude $c_1 = c_3 = 0$. At an interior point of edge $[v_k, v_{k+1}]$ neither $b_k$ nor $b_{k+1}$ vanishes. This implies that $c_2 = 0$. A similar calculation on each edge shows that all coefficients of non-diagonal terms in $Q$ are zero. \hfill \Box
C. The Map to $I(\mathcal{C})_2$

We define a surjective map onto $I(\mathcal{C})_2$. Computing the dimensions of the image and kernel is central to characterizing Wachspress quadratics. We first use the map to calculate the dimension of the vector space of polynomials in $I(\mathcal{C})_2$ that are supported on diagonal monomials. Later we argue that $I_2$ has the same dimension, implying that these vector spaces are equal, which yields the desired characterization of Wachspress quadratics.

Define the map $\Psi : \mathbb{C}[\mathbb{P}^\Delta]^3_1 \to I(\mathcal{C})_2$ by $F \mapsto F \cdot \tau$, where $\cdot$ is the scalar product.

**Lemma II.2.** The kernel of $\Psi$ is three-dimensional.

**Proof.** Let $\sigma_1, \sigma_2,$ and $\sigma_3$ be independent linear forms generating $I(\mathcal{C})$. Let $\mathbb{C}[\mathcal{C}]_2$ be the degree two piece of the coordinate ring of $\mathcal{C}$. Then we have $\dim(\mathbb{C}[\mathcal{C}]_2) = \binom{N-4+2}{2} = \binom{N-2}{2}$. To see this, note that $\mathcal{C}$ is projectively equivalent to coordinate plane cut out by the ideal $(x_{N-2}, x_{N-1}, x_N)$, so $\mathbb{C}[\mathcal{C}] \cong \mathbb{C}[x_1, \ldots, x_N]/(x_{N-2}, x_{N-1}, x_N) \cong \mathbb{C}[x_1, \ldots, x_{N-3}]$, and $\dim(\mathbb{C}[x_1, \ldots, x_{N-3}]_2) = \binom{N-2}{2}$. Now observe that $\dim(I(\mathcal{C})_2) = \dim(\mathbb{C}[\mathbb{P}^\Delta]_2) - \dim(\mathbb{C}[\mathcal{C}]_2) = \binom{N+1}{2} - \binom{N-2}{2} = 3N - 3$. Since any element of $I(\mathcal{C})_2$ is a combination of the three linear forms defining $I(\mathcal{C})$ with linear form coefficients, $\Psi$ is surjective thus we have $\ker(\Psi) = \dim(\mathbb{C}[\mathbb{P}^\Delta]_2^3) - I(\mathcal{C})_2 = 3N - (3N - 3) = 3$. 

Next, we examine the image of a vector under the map $\Psi$ and describe conditions so that this image is supported on diagonal monomials. Let $\mathcal{D}$ be the vector subspace of $\mathbb{C}[\mathbb{P}^\Delta]_2$ spanned by all diagonal monomials. If $w_i \in \mathbb{C}^3$ for $i = 1, \ldots, N,$

$$F = \sum_{i=1}^{N} x_i w_i$$
is an element of $\mathbb{C}[\mathbb{P}^\Delta]^3_1$. The projection $\tau$ is the triple:

$$\sum_{i=1}^N x_i v_i.$$ 

Thus,

$$\Psi(F) = F \cdot \tau = (\sum_{i=1}^N x_i w_i) \cdot (\sum_{i=1}^N x_i v_i) = \sum_{i,j=1}^N (w_i \cdot v_j + w_j \cdot v_i) x_i x_j.$$ 

For this image to be in $\mathcal{D}$ the coefficients of non-diagonal monomials must vanish;

$$w_i \cdot v_i = 0 \quad \text{and} \quad w_i \cdot v_{i+1} + w_{i+1} \cdot v_i = 0 \quad \text{for all } i.$$ 

Lemma II.3. The dimension of the vector space $\mathcal{D} \cap I(\mathcal{C})_2$ is $N - 3$.

Proof. We show the conditions in Equation (2.3) give $2N$ independent conditions on the $3N$-dimensional vector space $\mathbb{C}[\mathbb{P}^\Delta]^3_1$, and the solution space is $\Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2)$, thus $\dim(\Psi^{-1}(\mathcal{D} \cap I(\mathcal{C})_2)) = N$. The conditions are represented by the matrix equation:

$$\begin{pmatrix}
  v_1 \cdot w_1 \\
  \vdots \\
  v_N \cdot w_N \\
  v_1 \cdot w_2 + v_2 \cdot w_1 \\
  \vdots \\
  v_N \cdot w_1 + v_1 \cdot w_N 
\end{pmatrix} = 
\begin{pmatrix}
  v_1^T & 0 & \cdots & 0 \\
  0 & v_2^T & \vdots & \vdots \\
  \vdots & \vdots & \ddots & 0 \\
  0 & \cdots & 0 & v_N^T \\
  v_2^T & v_1^T & 0 \\
  0 & \cdots & \vdots & \vdots \\
  v_N^T & v_1^T & 0 & \cdots 
\end{pmatrix}
\begin{pmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_N 
\end{pmatrix} = 
\begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{pmatrix},$$

where the $v_i$ and $w_i$ are column vectors the superscript $T$ indicates transpose. The matrix $M$ in the middle is a $2N \times 3N$ matrix, and the proof will be complete if the rows are shown to be independent. Denote the rows of $M$ by $R_1, \ldots, R_N, R_{N+1} \ldots, R_{2N}$ and
let \( \gamma_1 R_1 + \cdots + \gamma_N R_N + \gamma_{N+1} R_{N+1} + \cdots + \gamma_{2N} R_{2N} \) be a dependence relation among the rows. The first three columns give the dependence relation \( \gamma_1 v_1 + \gamma_{N+1} v_2 + \gamma_{2N} v_N = 0 \). Since \( v_N, v_1, \) and \( v_2 \) are adjacent vertices of a polyhedral cone, they must be independent, so \( \gamma_1, \gamma_{N+1}, \) and \( \gamma_{2N} \) must be zero. Looking at the other columns we can similarly show the rest of the \( \gamma_i \)'s are zero.

Since the restriction \( \Psi : \Psi^{-1}(\mathcal{D} \cap \mathcal{I}(\mathcal{C}))_2 \to \mathcal{D} \cap \mathcal{I}(\mathcal{C})_2 \) remains surjective we deduce \( \dim(\mathcal{D} \cap \mathcal{I}(\mathcal{C})_2) = \dim(\Psi^{-1}(\mathcal{D} \cap \mathcal{I}(\mathcal{C})_2)) - \dim(\ker(\Psi)) = N - 3. \)

D. Wachspress Quadratics

We first compute the dimension of the space of Wachspress quadratics \( I_2 \). For this we study a surjective map with kernel \( I_2 \). Then we give an explicit set of quadratics that span \( I_2 \). Lastly, we prove that \( I_2 \) consists of the quadratic polynomials that are supported on diagonal monomials and vanish on the center of projection.

Set \( \gamma(i) := \{1, \ldots, N\} \setminus \{i - 1, i\} \) and \( \gamma(i, j) := \gamma(i) \cap \gamma(j) \). The image of a diagonal monomial \( x_i x_j \) under the pullback map \( \beta^* : \mathbb{C}[\mathbb{P}^\Delta] \to \mathbb{C}[\mathbb{P}^2] \) is

\[
bb = \alpha_i \alpha_j \prod_{k \in \gamma(i)} \ell_k \prod_{m \in \gamma(j)} \ell_m = \alpha_i \alpha_j \prod_{k=1}^{N} \ell_k \prod_{m \in \gamma(i, j)} \ell_m,
\]

and each has the common factor \( P := \prod_{k=1}^{N} \ell_k \). To find the quadratic relations among Wachspress coordinates it suffices to find linear relations among products \( \prod_{m \in \gamma(i, j)} \ell_m \in \mathbb{C}[\mathbb{P}^2]_{N-4} \) for diagonal pairs \( i, j \).

Define the map \( \phi : \mathcal{D} \to \mathbb{C}[\mathbb{P}^2]_{N-4} \) by

\[
x_i x_j \mapsto \frac{bb}{P},
\]

and extending by linearity. This is simply \( \beta^* \) restricted to \( \mathcal{D} \) and divided by \( P \). By Lemma II.1 it follows that \( I_2 = \ker(\phi) \subseteq \mathcal{D} \).
Example II.4. (Surjectivity of $\phi$ for the Hexagon)

For a hexagon we have $\phi : \mathcal{D} \cong \mathbb{C}^9 \to \mathbb{C}^6 \cong \mathbb{C}[\mathbb{P}^2]_2$. We show the image is six-dimensional by exhibiting six diagonal monomials with independent images. We consider the images of diagonal monomials not including $x_1$. Let $p_{ij} := \ell_i \cap \ell_j$ be the external vertex at the intersection of non-adjacent edges $\ell_i$ and $\ell_j$. The $(i,j)^{th}$ entry in Table I is the value of the image of the diagonal monomial in column $j$ at the external vertex in row $i$, a star $\ast$ represents a nonzero number, and a blank space is zero. The external vertices $p_{ij}$ used are those with $j \neq 6$, and they arranged with indices in lexicographic order down the rows. The lower triangular nature of Table I
demonstrates the linear independence of the images of the diagonal monomials that appear across the top row. Successively evaluating any dependence relation at the intersection points shows that all coefficients are zero.

The same holds for any polygon.

**Lemma II.5.** The map $\phi : \mathcal{D} \to \mathbb{C}[\mathbb{P}^2]_{N-4}$ is a surjective with $\dim(\ker \phi) = N - 3$. It follows that $\dim(I_2) = N - 3$. 

\begin{table}[h]
\centering
\begin{tabular}{cccccc}
\hline
& $x_2x_4$ & $x_2x_5$ & $x_2x_6$ & $x_3x_5$ & $x_3x_6$ & $x_4x_6$ \\
\hline
$p_{13}$ & $\ast$ \\
$p_{14}$ & $\ast$ & $\ast$ \\
$p_{15}$ & $\ast$ & $\ast$ \\
$p_{24}$ & $\ast$ & $\ast$ & $\ast$ \\
$p_{25}$ & $\ast$ & $\ast$ & $\ast$ & $\ast$ \\
$p_{35}$ & $\ast$ & $\ast$ & $\ast$ & $\ast$ & $\ast$ \\
\hline
\end{tabular}
\caption{Values of images of diagonal monomials at intersection points}
\end{table}
\textit{Proof.} There are $N - 3$ diagonal monomials that have $x_1$ as a factor. We show that the images of the remaining

$$N(N - 3)/2 - (N - 3) = (N - 3)(N - 2)/2 = \dim(\mathbb{C}[\mathbb{P}^2]_{N-4})$$

diagonal monomials are independent. Set $p_{ij} := \ell_i \cap \ell_j$. In Table II, a star, $\ast$, represents a nonzero number, a blank space is zero. The $(i, j)^{th}$ entry in Table II represents the value of the image of the diagonal monomial in column $j$ at the external vertex in row $i$. The external vertices not lying on $\ell_N$ are arranged down the rows with their indices in lexicographic order. The lower triangular nature of Table II

<table>
<thead>
<tr>
<th></th>
<th>$x_2x_4$</th>
<th>$\cdots$</th>
<th>$x_2x_N$</th>
<th>$x_3x_5$</th>
<th>$\cdots$</th>
<th>$x_3x_N$</th>
<th>$\cdots$</th>
<th>$x_{N-3}x_{N-1}$</th>
<th>$x_{N-3}x_N$</th>
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shows the independence of the images. We have found $\dim(\mathbb{C}[\mathbb{P}^2]_{N-4})$ independent images and hence $\phi$ is surjective. This is a map from a vector space of dimension $N(N - 3)/2$ to one of dimension $(N - 2)(N - 3)/2$. Since this map is surjective the
kernel has dimension $N(N - 3)/2 - (N - 2)(N - 3)/2 = N - 3$. \hfill \Box

There is a generating set for $I_2$ where each generator is a scalar product with the vector $\tau$. The other vectors in these scalar products are

$$\Lambda_k = \frac{x_{k+1}}{\alpha_{k+1}}n_{k+1} - \frac{x_k}{\alpha_k}n_{k-1} \in \mathbb{C}[\mathbb{P}^3_1].$$

**Lemma II.6.** The vectors $\{\Lambda_1, \ldots, \Lambda_N\}$ form a basis for the space $\Psi^{-1}(D \cap I(C)_2)$.

**Proof.** Suppose that $\sum_{k=1}^N c_k \Lambda_k = 0$ is a linear dependence relation among the $\Lambda_k$. The coefficient of a variable $x_k$ is

$$\frac{1}{\alpha_k} (c_{k-1}n_k - c_k n_{k-1}).$$

By the dependence relation this must be zero, which implies that $n_{k-1}$ and $n_k$ are scalar multiples. This is impossible since they are normal vectors of adjacent facets of a polyhedral cone. Hence, $c_{k-1} = c_k = 0$ for all $k$ which shows that the $\Lambda_k$ are independent.

In the proof of Lemma II.3 we showed that $\dim(\Psi^{-1}(D \cap I(C)_2)) = N$ and we have just shown $\dim(\langle \Lambda_k \mid k = 1, \ldots, N \rangle) = N$. We now show $\langle \Lambda_k \mid k = 1, \ldots, N \rangle \subseteq \Psi^{-1}(D \cap I(C)_2)$, which proves the result since two vector spaces of the same dimension with one contained in the other are equal. The conditions stated in Equation (2.3) are what is required for $\Lambda_k \in \mathbb{C}[\mathbb{P}^3_1]$ to lie in $\Psi^{-1}(D \cap I(C)_2)$. We show these conditions are satisfied for each $\Lambda_k$.

Set $w_i := 0$ if $i \neq k, k+1$, $w_i := -n_{k-1}/\alpha_k$ if $i = k$, and $w_i = n_{k+1}\alpha_{k+1}$ for each fixed $k$. Then

$$\Lambda_k = \frac{x_{k+1}}{\alpha_{k+1}}n_{k+1} - \frac{x_k}{\alpha_k}n_{k-1} = \sum_{i=1}^N w_i v_i.$$  

If $i \neq k, k+1$, then clearly $w_i \cdot v_i = 0$. Since $n_{k-1} \cdot v_k = n_{k+1} \cdot v_{k+1} = 0$, we obtain $w_i \cdot v_i = 0$ for each $i$. We now show that $w_i \cdot v_{i+1} + w_{i+1} \cdot v_i = 0$ holds for $i = k$. We
have
\[ -\frac{n_{k-1}}{\alpha_k} \cdot v_{i+1} + \frac{n_{k+1}}{\alpha_{k+1}} \cdot v_i = -\frac{v_{k-1} \times v_k \cdot v_{k+1}}{\alpha_k} + \frac{v_{k+1} \times v_{k+2} \cdot v_k}{\alpha_{k+1}} \]
\[ = -\frac{\det(v_{k-1}, v_k, v_{k+1})}{\alpha_k} + \frac{\det(v_{k+1}, v_{k+2}, v_k)}{\alpha_{k+1}} = 0, \]
as \(\alpha_j = \det(v_{j-1}, v_j, v_{j+1})\). Thus the \(w_i\) satisfy the conditions in Equations (2.3) and we conclude \(\Lambda_k \in \Psi^{-1}(\mathcal{D} \cap I(C)_2)\).

\[ \\Box \]

\textbf{Theorem II.7. (Characterization of Wachspress Quadratics)}

The Wachspress quadratics are characterized as the quadratic polynomials in \(\mathbb{C}[\mathbb{P}^\Delta]\) that are diagonally supported and vanish on the center of projection. Further, the quadratics \(Q_k = \Lambda_k \cdot \tau\) for \(k = 1, \ldots, N\) span \(I_2\).

\textit{Proof.} Let \(z\) be the vector \((z, x, y)\). By definition of Wachspress coordinates,
\[ \tau(\beta(z)) = \sum_{i=1}^{N} b_i(z) \cdot v_i = z \cdot \sum_{i=1}^{N} b_i(z). \]

We have
\[ \Lambda_k(\beta(z)) = \alpha_{k+1}n_{k-1}b_k - \alpha_kn_{k+1}b_{k+1} \]
\[ = \alpha_k\alpha_{k+1}(n_{k-1} \prod_{j \neq k-1, k} \ell_j - n_{k+1} \prod_{j \neq k, k+1} \ell_j) \]
\[ = \alpha_k\alpha_{k+1} \prod_{j \neq k-1, k, k+1} \ell_j (n_{k-1} \ell_{k+1} - n_{k+1} \ell_{k-1}) \]
\[ = P [n_{k-1} (n_{k+1} \cdot z) - n_{k+1} (n_{k-1} \cdot z)], \]
where \(P = \alpha_k\alpha_{k+1} \prod_{j \neq k-1, k, k+1} \ell_j\). Set \(\overline{P} := P \sum_{i=1}^{N} b_i(z)\). Then we have
\[ Q_k(\beta(z)) = \tau(\beta(z)) \cdot \Lambda_k(\beta(z)) \]
\[ = \overline{P} z \cdot [n_{k-1} (n_{k+1} \cdot z) - n_{k+1} (n_{k-1} \cdot z)] \]
= \mathcal{P} \left[ (\mathbf{n}_{k-1} \cdot \mathbf{z})(\mathbf{n}_{k+1} \cdot \mathbf{z}) - (\mathbf{n}_{k+1} \cdot \mathbf{z})(\mathbf{n}_{k-1} \cdot \mathbf{z}) \right] = 0.

We have just shown that \( Q_k \in I_2 \). By Lemma II.6 we know \( \langle \Lambda_k \rangle = \Psi^{-1}(D \cap I(C)_2) \). Observe that \( \langle Q_1, \ldots, Q_N \rangle = \Psi(\langle \Lambda_k \rangle) = D \cap I(C)_2 \). Thus we have \( \dim(\langle Q_1, \ldots, Q_N \rangle) = N - 3 \) and by Lemma II.5 \( \dim(I_2) = N - 3 \). Therefore, since \( \langle Q_1, \ldots, Q_N \rangle \subseteq I_2 \), we can conclude that \( \langle Q_1, \ldots, Q_N \rangle = I_2 = D \cap I(C)_2 \).

E. Irreducible Decomposition of \( V(\langle I_2 \rangle) \)

We describe the decomposition of \( V(\langle I_2 \rangle) \) into its irreducible components. First observe that the variety \( \mathcal{W} \) is irreducible because it is the closure of the image of an irreducible variety under a rational map. We show that if a point of \( V(\langle I_2 \rangle) \) does not lie in the linear space \( \mathcal{C} \), then it lies in \( \mathcal{W} \). We begin with some useful lemmas.

**Lemma II.8.** For any \( i, j, \) and \( k \) we have

\[
|\mathbf{n}_i \mathbf{n}_j \mathbf{n}_k| = |v_j v_k v_{k+1}| \cdot |v_i v_{i+1} v_{j+1}| - |v_{j+1} v_k v_{k+1}| \cdot |v_i v_{i+1} v_j|
\]

**Proof.** Apply the vector and scalar triple product formulas \( \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \) and \( |\mathbf{a} \mathbf{b} \mathbf{c}| = \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} \),

\[
|\mathbf{n}_i \mathbf{n}_j \mathbf{n}_k| = \mathbf{n}_i \times \mathbf{n}_j \cdot \mathbf{n}_k = (\mathbf{n}_i \times (v_j \times v_{j+1})) \cdot \mathbf{n}_k
= [v_j (\mathbf{n}_i \cdot v_{j+1}) - v_{j+1} (\mathbf{n}_i \cdot v_j)] \cdot \mathbf{n}_k
= (v_j \cdot \mathbf{n}_k)(\mathbf{n}_i \cdot v_{j+1}) - (v_{j+1} \cdot \mathbf{n}_k)(\mathbf{n}_i \cdot v_j)
= |v_j v_k v_{k+1}| \cdot |v_i v_{i+1} v_{j+1}| - |v_{j+1} v_k v_{k+1}| \cdot |v_i v_{i+1} v_j|.
\]

\( \square \)

**Corollary II.9.**

\[
|\mathbf{n}_i \mathbf{n}_j \mathbf{n}_{j+1}| = \alpha_{j+1}|v_i v_{i+1} v_{j+1}|
\]
Proof. This follows from Lemma II.8 and the definition of $\alpha_{j+1}$. \qed

Corollary II.10.

$$|n_{i-1} n_i n_{i+1}| = \alpha_i \alpha_{i+1}$$

Proof. This follows from Lemma II.8 and the definition of $\alpha_i$ and $\alpha_{i+1}$. \qed

Lemma II.11. Let $x = [x_1 : \cdots : x_N] \in V(\langle I_2 \rangle) \setminus C$. If $\tau(x)$ is a base point $p_{ij} = n_i \times n_j$, then $x$ lies on the exceptional line $\hat{p}_{ij}$ over $p_{ij}$.

Proof. Since indices are cyclic we assume that $i = 1$. Thus $\tau(x) = p_{1,j} = n_1 \times n_j$ for some $j \notin \{N, 1, 2\}$. The relation $q_1(x) = \Lambda_1 \cdot \tau(x) = \Lambda_1 \cdot (n_1 \times n_j) = 0$ yields

$$L_1 := x_2 n_2 \cdot p_{1,j} - x_1 n_N \cdot p_{1,j} = 0. \quad (2.4)$$

The relation $q_j(x) = 0$ implies,

$$L_j := x_{j+1} | n_{j+1} n_1 n_j | - x_j | n_2 n_1 n_j | = 0. \quad (2.5)$$

Also,

$$q_2(x) = (x_3 n_3 - x_2 n_1) \cdot n_1 \times n_j = x_3 | n_3 n_1 n_j | = 0,$$

implying $x_3 = 0$ since $| n_3 n_1 n_j | \neq 0$ if $j \neq 3$. Assume $x_k = 0$ for $3 \leq k < j - 1$.

Note that

$$q_k(x) = (x_{k+1} n_{k+1} - x_k n_{k-1}) \cdot n_1 \times n_j = x_{k+1} | n_{k+1} n_1 n_j | = 0,$$

hence $x_{k+1} = 0$ since $| n_{k+1} n_1 n_j | \neq 0$ and by induction $x_k = 0$ for $3 \leq k \leq j - 1$.

An analogous argument shows that $x_k = 0$ for $j + 2 \leq k \leq N$. Hence $x$ lies on the line $V(L_1, L_j, x_k \mid k \notin \{1, 2, j, j+1\})$, which is the exceptional line $\hat{p}_{1,j}$. \qed

Theorem II.12. The subset $V(\langle I_2 \rangle) \setminus C$ is contained in $W$. It follows that the variety $V(\langle I_2 \rangle)$ has irreducible decomposition $W \cup C$. 
Proof. Let \( x = [x_1 : \cdots : x_N] \in \mathbb{V}((I_2)) \setminus \mathcal{C} \). The Wachspress quadratics give the relations

\[
x_{r+1} \cdot n_{r+1} \tau = x_r \cdot n_{i-1} \cdot \tau
\]

(2.6)

for each \( r = 1, \ldots N \). Claim: For each \( k \in \{1, \ldots, N\} \) that \( b_k(\tau(x)) = A(\tau(x)) \cdot x_k \)

where the triangulation in Figure 8 is used to express the adjoint \( A \). By definition of the Wachspress map \( \beta \) we obtain

\[
\beta(\tau(x)) = A(\tau(x)) \cdot x.
\]

(2.7)

Provided \( A(\tau(x)) \neq 0 \), the result follows since \( \beta(\tau(x)) \in \mathbb{P}^\Delta \) is a nonzero scalar multiple of \( x \), hence \( x \) is in the image of the Wachspress map and thus lies on \( \mathcal{W} \). If \( x \in \mathbb{V}((I_2)) \setminus \mathcal{C} \) and \( A(\tau(x)) = 0 \), then by Equation (2.7) \( \beta(\tau(x)) = 0 \) and hence \( \tau(x) \) is a basepoint of \( \beta \). Thus \( \tau(x) = n_i \times n_j \) for some diagonal pair \( (i, j) \). By Lemma II.11 \( x \) lies on an exceptional line and hence lies on \( \mathcal{W} \).

We prove the claim. Since all indices are cyclic it suffices to assume \( k = 3 \). We introduce the notation: \( |n_{ijk}| := |n_i n_j n_k| = \det(n_i, n_j, n_k) \) and

\[
n_{i_1 \ldots i_m} \cdot \tau := \prod_{j=1}^{m} (n_{i_j} \cdot \tau).
\]

This is the product of \( m \) linear forms in the coordinates of \( \mathbb{P}^\Delta \), and with this notation

![Fig. 8. Triangulation used for adjoint](image)
\( b_3(\tau) = n_{1,4,5,...,N} \cdot \tau \). For each \( r \in \{3, \ldots, N\} \) define

\[
S_r : = (n_{4-r} \cdot \tau) n_1 \cdot \left[ \sum_{i=3}^{r} v_i (n_{r+1,...,N} \cdot \tau) x_i + \sum_{i=r+1}^{N} v_i (n_{r-1,...,i-2} \cdot \tau) (n_{i+1,...,N} \cdot \tau) x_r \right],
\]

where we set \( n_{i,...,j} \cdot \tau = 1 \) if \( j < i \). We show \( x A(\tau(x)) = S_3 = S_N = b_3(\tau(x)) \).

We first show that \( S_3 = x_3 A(\tau) \). Observe that \( x_3 A(\tau) \) is

\[
|n_{123}| (n_{4...N} \cdot \tau) x_3 + \sum_{i=1}^{N} |n_{1,i-1,i}| (n_{2,...,i-2} \cdot \tau) (n_{i+1,...,N} \cdot \tau) x_3,
\]

(2.8)

where we have expressed the adjoint \( A \) using the triangulation in Figure 8. Applying the scalar triple product to the determinants \(|n_{123}|\) and \(|n_{1,i-1,i}|\) in the expression (2.8),

\[
n_1 \cdot (n_2 \times n_3) (n_{4...N} \cdot \tau) x_3 + \sum_{i=1}^{N} n_1 \cdot (n_{i-1} \times n_i)(n_{2,...,i-2} \cdot \tau) (n_{i+1,...,N} \cdot \tau) x_3.
\]

(2.9)

Factoring an \( n_1 \) and noting that \( n_i \times n_{i+1} = v_{i+1} \), (2.9) becomes

\[
n_1 \left[ v_3 (n_{4...N} \cdot \tau) x_3 + \sum_{i=1}^{N} v_i (n_{2,...,i-2} \cdot \tau)(n_{i+1,...,N} \cdot \tau) x_3 \right] = S_3.
\]

Now we show \( S_N = b_3(\tau) \). Since \( n_{N+1,...,N} \cdot \tau = 1 \)

\[
S_N = (n_{4,...,N} \cdot \tau)n_1 \cdot \left( \sum_{i=3}^{N} v_i (n_{N+1,...,N} \cdot \tau)x_i \right) = (n_{4,...,N} \cdot \tau)n_1 \cdot \left( \sum_{i=3}^{N} v_i x_i \right).
\]

(2.10)

Observing that \( n_1 \cdot \sum_{i=1}^{2} x_i v_i = 0 \) we see that (2.10) is

\[
(n_{4,...,N} \cdot \tau) (n_1 \cdot \tau) = n_{1,4,...,N} \cdot \tau = b_3(\tau).
\]
We now claim that for $r \in \{3, \ldots, N - 1\}$ we have $S_r = S_{r+1}$. Indeed,

\[
S_r = (n_{4,\ldots,r} \cdot \tau) n_1 \cdot \left[ \sum_{i=3}^{r} v_i (n_{r+1,\ldots,N} \cdot \tau) x_i + \sum_{i=r+1}^{N} v_i (n_{r,\ldots,i-2} \cdot \tau) (n_{i+1,\ldots,N} \cdot \tau)(n_{r-1} \cdot \tau) x_i \right]
\]

where we have applied relation (2.6) to the last term. Next we factor out $n_{r+1} \cdot \tau$ to obtain

\[
(n_{4,\ldots,r+1} \cdot \tau) n_1 \cdot \left[ \sum_{i=3}^{r} v_i (n_{r+2,\ldots,N} \cdot \tau) x_i + \sum_{i=r+1}^{N} v_i (n_{r,\ldots,i-2} \cdot \tau) (n_{i+1,\ldots,N} \cdot \tau)(n_{r+1} \cdot \tau) x_{r+1} \right]
\]

Lastly, since the expressions in both summations agree at the index $i = r + 1$ we can shift the indices of summation,

\[
(n_{4,\ldots,r+1} \cdot \tau) n_1 \cdot \left[ \sum_{i=3}^{r+1} v_i (n_{r+2,\ldots,N} \cdot \tau) x_i + \sum_{i=r+2}^{N} v_i (n_{r,\ldots,i-2} \cdot \tau) (n_{i+1,\ldots,N} \cdot \tau) x_{r+1} \right],
\]

which is precisely $S_{r+1}$, proving the claim. The claim shows that $S_3 = S_N$, hence (2.7) holds and we conclude that $x$ lies in $\mathcal{W}$ if $A(\tau(x)) \neq 0$. \qed
CHAPTER III

THE WACHSPRESS CUBICS

A. Introduction

Theorem II.12 shows that the Wachspress quadratics do not suffice to cut out the Wachspress variety \( \mathcal{W} \). We now construct cubics, the \textit{Wachspress cubics}, that lie in the Wachspress ideal and are not contained in the ideal generated by the Wachspress quadratics \( I_2 \). These cubics have an elegant expression as the determinant of a \( 3 \times 3 \) matrix of linear forms. The proof that they lie in the ideal of \( \mathcal{W} \) uses the adjoint polynomial \( A \) of the dual polygon \( \Delta^* \) and that adjoints are independent of the triangulation used to express them, see Theorem I.18. In the next chapter we show that Wachspress quadratics and cubics cut out \( \mathcal{W} \) set-theoretically. The proof centers on the construction of several rational maps that are equivalent to the linear projection \( \tau \) on \( \mathcal{W} \). In the second half of this chapter, we construct these rational maps.

B. Construction of Wachspress Cubics

A cubic monomial \( x_i x_j x_k \) in \( \mathbb{C}[\mathbb{P}^\Delta] \) is a \( \Delta \)-\textit{monomial} if any pair of the variables \( x_i, x_j, x_k \) forms a diagonal monomial. The triple of indices of the variables in a \( \Delta \)-monomial is a \( \Delta \)-\textit{triple}. Identifying variable \( x_i \) with vertex \( v_i \), a \( \Delta \)-monomial is a triangle inscribed in \( \Delta \) formed by diagonals.

\textbf{Notation III.1.} The set \( \gamma(i) \) is \( \{1, \ldots, N\} \setminus \{i - 1, i\} \) and we set \( \gamma(i, j, k) := \gamma(i) \cap \gamma(j) \cap \gamma(k) \).

\textbf{Lemma III.2.} Evaluating a \( \Delta \)-monomial at Wachspress coordinates yields

\[ x_i x_j x_k(\beta) = b_i b_j b_k = P^2 \prod_{m \in \gamma(i,j,k)} \ell_m, \]

(3.1)
where $P$ is the product of all the linear forms $\ell_i$ defining the edges of $\Delta$.

Recall
\[
\Lambda_r = \frac{x_{r+1}}{\alpha_{r+1}} n_{r+1} - \frac{x_r}{\alpha_r} n_{r-1}
\]
as in Chapter 2, see Lemma II.6.

**Theorem III.3.** For a $\Delta$-triple $i, j, k$ the polynomial
\[
w_{i,j,k} := \det(\Lambda_i, \Lambda_j, \Lambda_k),
\]
lies in the Wachspress ideal.

**Example III.4.** For $N = 6$ there are two $\Delta$-triples $(1, 3, 5)$ and $(2, 4, 6)$ and hence we obtain the two cubics $w_{1,3,5}$ and $w_{2,4,6}$.

The cubics $w_{i,j,k}$ will be referred to as *Wachspress cubics*. Before taking on the proof of Theorem III.3 let us first perform a preliminary calculation and make some observations. There are no triangular triples if $N < 6$; hence, there are no Wachspress cubics for such $N$. Thus when discussing $\Delta$-triples we are implicitly assuming $N \geq 6$.

**Remark III.5.** By making a change of variable, replacing $x_i$ with $x_i/\alpha_i$, we ignore the constants $\alpha_i$ in what follows.

**Preliminary Calculation III.6.** Using the definition of the $\Lambda$’s and the multilinearity of determinant,
\[
det(\Lambda_i, \Lambda_j, \Lambda_k) = |n_{i+1} n_{j+1} n_{k+1} x_{i+1} x_{j+1} x_{k+1} - n_{i+1} n_{j+1} n_{k-1} x_{i+1} x_{j+1} x_k - n_{i+1} n_{j-1} n_{k+1} x_{i+1} x_j x_{k+1} + n_{i+1} n_{j-1} n_{k-1} x_{i+1} x_j x_k - n_{i-1} n_{j+1} n_{k+1} x_i x_{j+1} x_{k+1} + n_{i-1} n_{j+1} n_{k-1} x_i x_{j+1} x_k + n_{i-1} n_{j-1} n_{k+1} x_i x_j x_{k+1} - n_{i-1} n_{j-1} n_{k-1} x_i x_j x_k|.
\]
Proof. The three equations \(i + 1 = j - 1\), \(j + 1 = k - 1\), and \(k + 1 = i - 1\) involving the indices of our \(\Delta\)-triple yield four cases:

1. All three hold  
2. Two hold  
3. One holds  
4. None hold.

We prove Theorem III.3 in each of these four cases separately.

Case 1: The \(\Delta\)-triple \((i, j, k)\) satisfies Case 1 if and only if \(N = 6\). For \(N = 6\) there are only two \(\Delta\)-triples; \((1, 3, 5)\) and \((2, 4, 6)\), hence \(w_{1,3,5}\) and \(w_{2,4,6}\) are the only Wachspress cubics. We show that \(w_{1,3,5}\) vanishes on Wachspress coordinates. The case of \(w_{2,4,6}\) is similar. All but two of the determinants in Preliminary Calculation III.6 vanish, leaving

\[
w_{1,3,5} = \det(\Lambda_1, \Lambda_3, \Lambda_5) = |n_2 \ n_4 \ n_6|x_{2}x_{4}x_{6} - |n_6 \ n_2 \ n_4| x_{1}x_{3}x_{5}. \tag{3.3}
\]

Notice that the coefficients are equal, thus we finish the proof by showing that \(x_{1}x_{3}x_{5} - x_{2}x_{4}x_{6}\) vanishes on Wachspress coordinates. The monomials \(x_{1}x_{3}x_{5}\) and \(x_{2}x_{4}x_{6}\) evaluated at Wachspress coordinates are \(b_{1}b_{3}b_{5}\) and \(b_{2}b_{4}b_{6}\), respectively. Using Lemma III.2 we observe that

\[
b_{1}b_{3}b_{5} = \prod_{m \in \gamma(1,3,5)} \ell_{m} = \prod_{m \in \gamma(2,4,6)} \ell_{m} = b_{2}b_{4}b_{6},
\]

hence \(x_{1}x_{3}x_{5} - x_{2}x_{4}x_{6}\) vanishes on Wachspress coordinates.

Case 2: We can assume without loss of generality \(i + 1 \neq j - 1\), \(j + 1 = k - 1\), and \(k + 1 = i - 1\). Four coefficients vanish in the Preliminary calculation, yielding

\[
w_{ijk} = |n_{i+1} \ n_{j+1} \ n_{i-1}|x_{i+1}x_{j+1}x_{i-1} - |n_{i+1} \ n_{j-1} \ n_{i-1}|x_{i+1}x_{j}x_{i-1} \\
+ |n_{i+1} \ n_{j+1} \ n_{i+1}|x_{i+1}x_{j}x_{i-2} - |n_{i-1} \ n_{j-1} \ n_{j+1}|x_{i}x_{j}x_{i-2}.
\]
Evaluating this at Wachspress coordinates yields,

\[ w_{ijk} \circ \beta = n_{i+1}n_{j+1}n_{i-1} \prod_{m \in \gamma(i+1,j+1,i-1)} \ell_m + n_{i+1}n_{j-1}n_{i-1} \prod_{m \in \gamma(i+1,j,i-1)} \ell_m - n_{i+1}n_{j-1}n_{j+1} \prod_{m \in \gamma(i,j,i-1)} \ell_m - n_{i-1}n_{j-1}n_{j+1} \prod_{m \in \gamma(i,j,j+1)} \ell_m \]

\[ = P^2 \left( \prod_{m \in \gamma(i-1,i+1,j+1,j)} \ell_m \right) \left[ n_{i+1}n_{j+1}n_{i-1} \ell_{j-1} - n_{i+1}n_{j-1}n_{i-1} \ell_{j+1} + n_{i+1}n_{j-1}n_{j+1} \ell_{i-1} - n_{i-1}n_{j-1}n_{j+1} \ell_{i+1} \right] - \left[ n_{i+1}n_{j-1}n_{i-1} \ell_{j+1} + n_{i+1}n_{j+1}n_{j-1} \ell_{i-1} \right] \]

where \( P = \prod_{i=1}^{N} \ell_i \). The last factor is the difference of the two adjoints respect to the

\[ \text{Fig. 9. Case 2 triangulation} \]

triangulations of the quadrilateral in Figure 9.

**Case 3:** Assume without loss of generality \( i+1 \neq j-1, j+1 \neq k-1, \) and \( k+1 = i-1 \).

In this case two coefficients vanish in the Preliminary calculation and after evaluating
at Wachspress coordinates we obtain,

\[ w_{ijk} \circ \beta = |n_{i+1}n_{j+1}n_{i-1}| \prod_{m \in \gamma(i+1,j+1,k+1)} \ell_m - |n_{i+1}n_{j+1}n_{k-1}| \prod_{m \in \gamma(i+1,j+1,k)} \ell_m - \\
|n_{i+1}n_{j-1}n_{i-1}| \prod_{m \in \gamma(i+1,j,k+1)} \ell_m + |n_{i+1}n_{j-1}n_{k-1}| \prod_{m \in \gamma(i+1,j,k)} \ell_m + \\
|n_{i-1}n_{j+1}n_{k-1}| \prod_{m \in \gamma(i,j+1,k+1)} \ell_m - |n_{i-1}n_{j+1}n_{k-1}| \prod_{m \in \gamma(i,j,k)} \ell_m \]

\[ = p^2 \left( \prod_{m \in \gamma(i,j,k+1), \ell_m} \ell_m \right) \left( |n_{i+1}n_{j+1}n_{i-1}| \ell_{i-1} \ell_{j-1} - \\
|n_{i+1}n_{j+1}n_{k-1}| \ell_{i-1} \ell_{j-1} - |n_{i+1}n_{j-1}n_{i-1}| \ell_{j-1} \ell_{k-1} + \\
|n_{i+1}n_{j-1}n_{k-1}| \ell_{j-1} \ell_{i-1} + |n_{i-1}n_{j+1}n_{k-1}| \ell_{j+1} \ell_{i-1} - \\
|n_{i-1}n_{j+1}n_{k-1}| \ell_{j+1} \ell_{j+1} \right) \]

The last factor is the difference of adjoints with respect to the triangulations of the pentagon in Figure 10.

**Case 4:** In this case evaluation at Wachspress coordinates yields,

\[ w_{ijk} \circ \beta = |n_{i+1}n_{j+1}n_{k+1}| \prod_{m \in \gamma(i+1,j+1,k+1)} \ell_m - |n_{i+1}n_{j+1}n_{k-1}| \prod_{m \in \gamma(i+1,j+1,k)} \ell_m - \\
|n_{i+1}n_{j-1}n_{k+1}| \prod_{m \in \gamma(i+1,j,k+1)} \ell_m + |n_{i+1}n_{j-1}n_{k-1}| \prod_{m \in \gamma(i+1,j,k)} \ell_m + \\
|n_{i-1}n_{j+1}n_{k+1}| \prod_{m \in \gamma(i,j+1,k+1)} \ell_m + |n_{i-1}n_{j+1}n_{k-1}| \prod_{m \in \gamma(i,j,k)} \ell_m + \]

Fig. 10. Case 3 triangulation
\[ |n_{i-1}n_{j-1}n_{k+1}| \prod_{m \in \gamma(i,j,k+1)} \ell_m - |n_{i-1}n_{j-1}n_{k-1}| \prod_{m \in \gamma(i,j,k)} \ell_m \]

\[ = P^2 \left( \prod_{m \in \gamma(i,j,k)_{i+1,j+1,k+1}} \ell_m \right) \left( |n_{i+1}n_{j+1}n_{k+1}| \ell_{i-1} \ell_{j-1} \ell_{k-1} - \right. \]

\[ |n_{i+1}n_{j+1}n_{k-1}| \ell_{i-1} \ell_{j-1} \ell_{k+1} - |n_{i+1}n_{j-1}n_{k+1}| \ell_{i-1} \ell_{j+1} \ell_{k-1} + \]

\[ |n_{i+1}n_{j-1}n_{k-1}| \ell_{i-1} \ell_{j+1} \ell_{k+1} - |n_{i-1}n_{j+1}n_{k+1}| \ell_{i+1} \ell_{j-1} \ell_{k-1} + \]

\[ |n_{i-1}n_{j+1}n_{k+1}| \ell_{j+1} \ell_{i-1} \ell_{k+1} + |n_{i-1}n_{j-1}n_{k+1}| \ell_{i+1} \ell_{j+1} \ell_{k-1} - \]

\[ |n_{i-1}n_{j-1}n_{k-1}| \ell_{i+1} \ell_{j+1} \ell_{k+1} \]

The last factor is the difference of adjoints expressed using the triangulations of the hexagon in Figure 11.

Fig. 11. Case 4 triangulation

C. The Approach for Obtaining a Set-Theoretic Result

Let \( \hat{I} \) be the ideal generated by the Wachspress quadratics and cubics and \( \mathbb{T}^\Delta \) the algebraic torus in \( \mathbb{P}^\Delta \). By construction we know that \( \mathcal{W} \subseteq \mathbb{V}(\hat{I}) \). To obtain our set-theoretic result, Theorem IV.4, we must show that \( \mathbb{V}(\hat{I}) \subseteq \mathcal{W} \). We have already shown in Chapter 2 that \( \mathbb{V}(\hat{I}) \setminus \mathcal{C} \subseteq \mathcal{W} \). Showing that \( \mathbb{V}(\hat{I}) \cap \mathbb{T}^\Delta \subseteq \mathcal{W} \) will be treated in Section A of Chapter 4. In the next section we learn how to deal with points that lie in the center \( \mathcal{C} \) and we will be able conclude that \( (\mathbb{V}(\hat{I}) \cap \mathcal{C}) \setminus \mathbb{T}^\Delta \subseteq \mathcal{W} \).
D. Another Expression for the Projection $\tau$ When $N$ is Odd

Our goal is to construct rational maps that are equivalent to the linear projection $\tau$ on the Wachspress variety $\mathcal{W}$. The construction of the maps differs slightly depending on the parity of the number of edges $N$ of the polygon $\Delta$. We first direct our attention to the odd case. Let $\Delta$ be an $N$-sided polygon with $N = 2k + 1$. We now define monomials that we be used to construct the projections.

**Definition III.7.** Define the monomial

$$M_i = \prod_{j=1}^{k} x_{i+2j}.$$  

For example, with $k = 4$, the monomial $M_1$ is $x_3x_5x_7x_9$ and $M_2$ is $x_1x_4x_6x_8$. The essential property of these monomials is revealed when we evaluate them at Wachspress coordinates.

**Lemma III.8.** The monomial $M_i$ evaluated at Wachspress coordinates $(b_1, \ldots, b_N)$ is $P^{k-1} \ell_i$ where $P = \prod_{j=1}^{N} \ell_j$.

**Proof.** Follows directly by evaluating $M_i(b_1, \ldots, b_N)$. \hfill $\square$

We are ready to define a collection of maps that are equivalent to the linear projection on $\mathcal{W}$. For $i = 1, \ldots, N$ define the rational map $\sigma_i : \mathbb{P}^\Delta \rightarrow \mathbb{P}^2$ by

$$\sigma_i := \frac{(n_i \times n_{i+1})}{A_i} M_{i-1} + \frac{(n_{i+1} \times n_{i-1})}{A_i} M_i + \frac{(n_{i-1} \times n_i)}{A_i} M_{i+1},$$

where $A_i = |n_{i-1} n_i n_{i+1}|$. The indeterminacy locus of $\sigma_i$ is $\mathbb{V}(M_{i-1}, M_i, M_{i+1}) \subseteq \mathbb{P}^\Delta \setminus \mathbb{T}^\Delta$.

**Theorem III.9.** The map $\sigma_i$ is equivalent to $\tau$ on $\mathcal{W}$. 
that the same holds for $\sigma$. By Lemma III.8,

$$
\sigma_i(\beta(z)) = \frac{n_i \times n_{i+1}}{A_i} M_{i-1}(\beta(z)) + \frac{n_{i+1} \times n_{i-1}}{A_i} M_i(\beta(z)) + \frac{n_{i-1} \times n_i}{A_i} M_{i+1}(\beta(z))
$$

$$
= \frac{p^{k-1}}{A_i} \left( (n_i \times n_{i+1}) \ell_{i-1}(z) + (n_{i+1} \times n_{i-1}) \ell_i(z) + (n_{i-1} \times n_i) \ell_{i+1}(z) \right).
$$

We claim that $(n_i \times n_{i+1}) \ell_{i-1}(z) + (n_{i+1} \times n_{i-1}) \ell_i(z) + (n_{i-1} \times n_i) \ell_{i+1}(z) = A_i z$. We prove the claim by showing that

$$(n_i \times n_{i+1}) \ell_{i-1}(z) + (n_{i+1} \times n_{i-1}) \ell_i(z) + (n_{i-1} \times n_i) \ell_{i+1}(z) \cdot e_j = A_i z_j \quad (3.4)$$

for the standard basis vectors $e_1 = [1, 0, 0]$, $e_2 = [0, 1, 1]$, and $e_3 = [0, 0, 1]$. Observe that the left hand side of Equation 3.4 is

$$(n_i \times n_{i+1}) \ell_{i-1}(z) + (n_{i+1} \times n_{i-1}) \ell_i(z) + (n_{i-1} \times n_i) \ell_{i+1}(z) \cdot e_j$$

$$= |n_i n_{i+1} e_j| n_{i-1} \cdot z + |n_{i+1} n_{i-1} e_j| n_i \cdot z + |n_{i-1} n_i e_j| n_{i+1} \cdot z$$

$$= (|n_i n_{i+1} e_j| n_{i-1} + |n_{i+1} n_{i-1} e_j| n_i + |n_{i-1} n_i e_j| n_{i+1}) \cdot z. \quad (3.5)$$

By applying Lemma I.18 we see that Equation 3.5 is $|n_{i-1} n_i n_{i+1}| e_j \cdot z = A_i z_j$, proving the claim. We have shown that the values of $\tau$ and $\sigma_i$ agree on the open set $W \setminus (V(P) \cup C)$, thus they are equivalent on $W$. \qed

**Lemma III.10.** The polynomials

$$
d_i := |n_i n_{i+1} n_{i+2}| M_{i-1} - |n_{i-1} n_i n_{i+2}| M_i + |n_{i-1} n_i n_{i+2}| M_{i+1} - |n_{i-1} n_i n_{i+1}| M_{i+2}
$$

for $i = 1, \ldots, N$ vanish on $W$. 

Proof. We evaluate $d_i$ at Wachspress coordinates:

$$d_i(\beta) = |n_i n_{i+1} n_{i+2}| M_{i-1}(\beta) - |n_{i-1} n_{i+1} n_{i+2}| M_i(\beta) + |n_{i-1} n_i n_{i+2}| M_{i+1}(\beta) - |n_{i-1} n_i n_{i+1}| M_{i+2}(\beta)$$

$$= P^{k-1}(|n_i n_{i+1} n_{i+2}| \ell_i - |n_{i-1} n_{i+1} n_{i+2}| \ell_i + |n_{i-1} n_i n_{i+2}| \ell_{i+1} - |n_{i-1} n_i n_{i+1}| \ell_{i+2})$$

$$= P^{k-1}(|n_i n_{i+1} n_{i+2}| \ell_i - |n_{i-1} n_i n_{i+2}| \ell_{i+1} - (|n_{i-1} n_{i+1} n_{i+2}| \ell_i + |n_{i-1} n_i n_{i+1}| \ell_{i+2}))$$

$$= P^{k-1}(|n_i n_{i+1} n_{i+2}| n_{i-1} + |n_{i-1} n_i n_{i+2}| n_{i+1} - |n_{i-1} n_i n_{i+1} n_{i+2}| n_i - |n_{i-1} n_i n_{i+1} n_{i+2}| \cdot z).$$

By Lemma I.19 the factor in parentheses in the last line is zero. \qed

Let $J$ be the ideal generated by the Wachspress quadratics, Wachspress cubics, and the $d_i$.

**Conjecture III.11.** The polynomials $d_i$ lie in the ideal $\hat{I}$; hence, $J = \hat{I}$.

**Lemma III.12.** The rational maps $\sigma_1, \ldots, \sigma_N$ are equivalent on $\forall(J)$.

**Proof.** It suffices to show that $\sigma_i \cong \sigma_{i+1}$ modulo $J$ for any $i = 1, \ldots, N$. We show that the difference $\sigma_i - \sigma_{i+1}$ is zero modulo $J$.

$$\sigma_i - \sigma_{i+1} = |n_i n_{i+1} n_{i+2}| M_{i-1} - |n_i n_{i+1} n_{i+2}| M_{i+2}$$

$$+ (|n_i n_{i+1} n_{i+2}| n_{i+1} n_{i-1} - |n_{i-1} n_i n_{i+1}| n_{i+1} n_{i+1} n_{i+1} M_i +$$

$$+ (|n_i n_{i+1} n_{i+2}| n_{i+1} n_{i-1} - |n_{i-1} n_i n_{i+1}| n_{i+2} n_i) M_{i+1}$$

$$= |n_i n_{i+1} n_{i+2}| M_{i-1} - |n_i n_{i+1} n_{i+2}| M_{i+2}$$

$$n_{i+1} \times (|n_i n_{i+1} n_{i+2}| n_{i-1} - |n_{i-1} n_i n_{i+1}| n_{i+2}) M_i$$
\[
\left( \left| n_i \ n_{i+1} \ n_{i+2} \right| n_{i-1} - \left| n_{i-1} \ n_i \ n_{i+1} \right| n_{i+2} \right) \times n_i \ M_{i+1}.
\]

Notice that the two factors enclosed in parenthesis above are the same and by Lemma I.19 are equal to \( \left| n_{i-1} \ n_{i+1} \ n_{i+2} \right| n_i - \left| n_{i-1} \ n_i \ n_{i+2} \right| n_{i+1} \). Thus we have

\[
\sigma_i - \sigma_{i+1} = (n_i \times n_{i+1}) \left( \left| n_i \ n_{i+1} \ n_{i+2} \right| M_{i-1} - \left| n_{i-1} \ n_i \ n_{i+2} \right| M_i \\
+ \left| n_{i-1} \ n_i \ n_{i+2} \right| M_{i+1} - \left| n_{i-1} \ n_i \ n_{i+1} \right| M_{i+2} \right) = (n_i \times n_{i+1})d_i.
\]

Thus the maps \( \sigma_i \) and \( \sigma_{i+1} \) agree on \( J \) since \( d_i \in J \). Note that we have actually shown that \( A_i \sigma_i = A_{i+1} \sigma_{i+1} \) and more generally it follows that \( A_i \sigma_j = A_j \sigma_j \).

**Remark III.13.** We obtained in the proof of the preceding lemma the equation \( A_i \sigma_i = A_j \sigma_j \); however, we will assume we have \( \sigma_i = \sigma_j \), ignoring the constant \( A_i \) to simplify future arguments. The constants can be carried along without effecting our arguments but make for tedious bookkeeping.

**Lemma III.14.** For any indices \( i, j \in \{1, \ldots, N\} \) we have \( \ell_j(\sigma_i) = M_j \).

*Proof.* It is immediate from the definition of \( \sigma_i \) that \( \ell_i(\sigma_i) = M_i \). By Lemma III.12, on \( \mathbb{V}(J) \), we have

\[
\ell_j(\sigma_i) = \ell_j(\sigma_j) = M_j.
\]

**Lemma III.15.** For \( x = [x_1 : \cdots : x_N] \in \mathbb{V}(J) \) and for any \( i \in \{1, \ldots, N\} \),

\[
\beta \circ \sigma_i(x) = \left( \prod_{s=1}^{N} x_s \right)^{k-1} x.
\]

*Proof.* It suffices to show that for any \( j \in \{1, \ldots, N\} \),

\[
(b_j \circ \sigma_i)(x) = \left( \prod_{s=1}^{N} x_s \right)^{k-1} x_j.
\]
Observe,

$$(b_j \circ \sigma_i)(x) = \prod_{s \neq j-1, j} \ell_s(\sigma_i(x)) = \prod_{s \neq j-1, j} M_s = (\prod_{s=1}^{N} x_s)^{k-1} x_j.$$ 

To see the last equality it suffices to set $j = 1$. Observe $M_{i-1}M_i = \prod_{j \neq i} x_j$ for any $j$; hence,

$$(b_1 \circ \sigma_i)(x) = \prod_{s \neq 1, N} M_s = (M_2M_3)(M_4M_5)\cdots(M_{N-3}M_{N-2})M_{N-1}$$

$$= (\prod_{j \neq 3} x_j)(\prod_{j \neq 5} x_j)\cdots (\prod_{j \neq N-2} x_j)x_1x_3\cdots x_{N-2} = P^{k-1}x_1.$$ 

\[\square\]

E. Another Expression for the Projection $\tau$ When $N$ is Even

We now find maps analogous to the $\sigma_i$’s in the case where the polygon $\Delta$ has an even number of edges. Let $N = 2k$ be even. Define the monomials $M_{i,j}$ for $1 \leq i, j \leq N$ with $i$ and $j$ of opposite parity as

$$M_{i,j} = x_{i-1} \prod_{m=1}^{\frac{j-i-1}{2}} x_{i+2m} \prod_{m=1}^{\frac{N-j+i-2}{2}} x_{j+2m} \quad (3.6)$$

if $i < j$ and $j - i > 1$, and

$$M_{i,i+1} = x_{i-1} \prod_{m=1}^{\frac{k-2}{2}} x_{i+2m+1}. \quad (3.7)$$

**Example III.16.** Let $k = 4$ and hence $N = 8$. Then,

$$M_{1,2} = x_4x_6x_8 \quad M_{1,6} = x_3x_5x_8 \quad M_{2,3} = x_1x_5x_7 \quad M_{2,7} = x_1x_4x_6$$

$$M_{1,4} = x_3x_6x_8 \quad M_{1,8} = x_3x_5x_7 \quad M_{2,5} = x_1x_4x_7,$$

and with $k = 5$, ...
\[
M_{1,2} = x_4x_6x_8x_{10} \quad M_{1,6} = x_3x_5x_8x_{10} \quad M_{2,3} = x_1x_5x_7x_9 \quad M_{2,7} = x_1x_4x_6x_9
\]
\[
M_{1,4} = x_3x_6x_8x_{10} \quad M_{1,8} = x_3x_5x_7x_{10} \quad M_{2,5} = x_1x_4x_7x_9 \quad M_{2,9} = x_1x_4x_6x_8
\]

Notice that for \(i\) even (or odd) \(M_{i-1,i}\) is the product of all even (or odd) indexed variables except \(x_i\). The important fact about these monomials is that under evaluation at Wachspress coordinates \(M_{i,j}\) is \(P_{k-2}^{i,j}\). We can now define maps analogous to the \(\sigma_i\)’s. For \(1 \leq i \leq N\) define the rational maps

\[
\zeta_i(x) := \left(\frac{n_{i+3} \times n_{i+5}}{\hat{A}_i}M_{i,i+1} + \frac{(n_{i+5} \times n_{i+1})}{\hat{A}_i}M_{i,i+3} + \frac{(n_{i+1} \times n_{i+3})}{\hat{A}_i}M_{i,i+5},\right.
\]

where \(\hat{A}_i = |n_{i+1} n_{i+3} n_{i+5}|\).

**Remark III.17.** Observe that \(\ell_i(\zeta_{i-1}(x)) = M_{i-1,i}\).

The polynomials in Lemma III.18 are completely analogous to the polynomials \(d_i\) in Lemma III.10 in the odd case, so we also denote them by \(d_i\).

**Lemma III.18.** The following polynomials \(d_i\) for \(i = 1, \ldots, N\) vanish on \(W\).

\[
d_i := n_{i+3}n_{i+5}n_{i+7}|x_{i+1}M_{i,i+1} - n_{i+1}n_{i+3}n_{i+5}|x_{i+2}M_{i+2,i+7} -
\]
\[
|n_{i+1}n_{i+3}n_{i+5}|x_{i+1}M_{i,i+3} + n_{i+1}n_{i+3}n_{i+5}|x_{i+1}M_{i+1,i+5}.
\]

**Proof.** The proof proceeds in the same manner as the proof of Lemma III.10 and is omitted. \qed

**Conjecture III.19.** The polynomials \(d_i\) for \(i = 1, \ldots, N\) lie in \(\hat{I}\).

Let \(J\) be the ideal generated by Wachspress quadratics, Wachspress cubics, and the \(d_i\).

**Lemma III.20.** For \(1 \leq i \leq N\) the rational maps \(\zeta_i\) and \(\zeta_{i+2}\) are equivalent on the variety \(V(J)\). Further, we have \(\hat{A}_i x_{i+1} \zeta_i = \hat{A}_{i+2} x_{i+2} \zeta_{i+2}\) modulo \(J\).
Proof. We show that the difference \( \hat{A}_{i+2} x_{i+1} \zeta_i - \hat{A}_i x_{i+2} \zeta_{i+2} \) is zero modulo \( J \).

\[
\hat{A}_{i+2} x_{i+1} \zeta_i - \hat{A}_i x_{i+2} \zeta_{i+2} = \left| n_{i+3} n_{i+5} n_{i+7} (n_{i+3} \times n_{i+5}) x_{i+1} M_{i,i+1} \right| + |n_{i+3} n_{i+5} n_{i+7} (n_{i+5} \times n_{i+1}) M_{i,i+3} + \\
|n_{i+3} n_{i+5} n_{i+7} (n_{i+1} \times n_{i+3}) x_{i+1} M_{i+1,i+5} - |n_{i+1} n_{i+3} n_{i+5} (n_{i+5} \times n_{i+7}) x_{i+2} M_{i+2,i+3} - \\
|n_{i+1} n_{i+3} n_{i+5} (n_{i+7} \times n_{i+3}) x_{i+2} M_{i+2,i+5} - \\
|n_{i+1} n_{i+3} n_{i+5} (n_{i+3} \times n_{i+5}) x_{i+2} M_{i+2,i+7}
\]

(3.8)

It is not difficult to check directly from the definitions that \( x_{i+1} M_{i,i+3} = x_{i+2} M_{i+2,i+3} \) and \( x_{i+1} M_{i+1,i+5} = x_{i+2} M_{i+2,i+5} \). Using this we can combine term in (3.8) to obtain

\[
|n_{i+3} n_{i+5} n_{i+7} (n_{i+3} \times n_{i+5}) x_{i+1} M_{i,i+1} - |n_{i+1} n_{i+3} n_{i+5} (n_{i+3} \times n_{i+5}) x_{i+2} M_{i+2,i+7} + \\
(|n_{i+3} n_{i+5} n_{i+7} (n_{i+5} \times n_{i+1}) - |n_{i+1} n_{i+3} n_{i+5} (n_{i+5} \times n_{i+7}) x_{i+1} M_{i,i+3} + \\
(|n_{i+3} n_{i+5} n_{i+7} x_{i+1} M_{i+1,i+5} - |n_{i+1} n_{i+3} n_{i+5} x_{i+2} M_{i+2,i+7} + \\
|n_{i+3} n_{i+5} n_{i+7} n_{i+1} - |n_{i+1} n_{i+3} n_{i+5} n_{i+7} x_{i+1} M_{i,i+3} + \\
(|n_{i+3} n_{i+5} n_{i+7} n_{i+1} - |n_{i+1} n_{i+3} n_{i+5} n_{i+7}) x_{i+3} x_{i+1} M_{i+1,i+5}.
\]

The two factors in parentheses are the same and by Lemma I.19 are both equal to \( |n_{i+1} n_{i+3} n_{i+5} |n_{i+5} - |n_{i+1} n_{i+5} n_{i+7} |n_{i+3} \). The last line in the Equation above becomes,

\[
|n_{i+3} n_{i+5} n_{i+7} (n_{i+3} \times n_{i+5}) x_{i+1} M_{i,i+1} - |n_{i+1} n_{i+3} n_{i+5} (n_{i+3} \times n_{i+5}) x_{i+2} M_{i+2,i+7} + \\
n_{i+5} \times (|n_{i+1} n_{i+3} n_{i+5} n_{i+5} - |n_{i+1} n_{i+5} n_{i+7} n_{i+3}) x_{i+1} M_{i,i+3} + \\
(|n_{i+1} n_{i+3} n_{i+5} n_{i+5} - |n_{i+1} n_{i+5} n_{i+7} n_{i+3} \times n_{i+3} x_{i+1} M_{i+1,i+5} = (n_{i+3} \times n_{i+5}) (|n_{i+3} n_{i+5} n_{i+7} x_{i+1} M_{i,i+1} - |n_{i+1} n_{i+3} n_{i+5} x_{i+2} M_{i+2,i+7} - \\
|n_{i+1} n_{i+5} n_{i+7} x_{i+1} M_{i,i+3} + |n_{i+1} n_{i+3} n_{i+5} x_{i+1} M_{i+1,i+5}).
\]
It remains to show that the maps $\zeta_i$ and $\zeta_j$ are equivalent for indices $i$ and $j$ of opposite parity. To show this it suffices to let $i = 1$ and $j = 2$. The rational maps $\zeta_1$ and $\zeta_2$ are represented by triples of forms of degree $k - 1$ in $\mathbb{C}[\mathbb{P}^\Delta]$. Let $\zeta_1 := [f_1 : f_2 : f_3], \zeta_2 = [g_1 : g_2 : g_3]$, and let $M_{fg}$ be the matrix

$$
\begin{bmatrix}
    f_1 & f_2 & f_3 \\
    g_1 & g_2 & g_3
\end{bmatrix}.
$$

**Lemma III.21.** The minors of the matrix $M_{fg}$ vanish on $W$.

*Proof.* The matrix $M_{fg}$ is simply the $2 \times 3$ matrix with rows $\zeta_1$ and $\zeta_2$. By definition, evaluation at Wachspress coordinates yields a matrix with rows $\zeta_1(\beta(z)) = Pz$ and $\zeta_2(\beta(z))$ for any $z \in \mathbb{P}^2$ where $P$ is the product of all the linear forms defining the edges of $\Delta$. This matrix clearly has rank one for all $z$, hence each minor of $M_{fg}$ vanishes on Wachspress coordinates so vanishes on $W$. \qed

**Lemma III.22.** The rational maps $\zeta_1$ and $\zeta_2$ are equivalent modulo the ideal generated by the minors of the matrix $M_{fg}$. Thus modulo these minors the rational maps differ by a rational function $c(x)$; i.e., $\zeta_2(x) = c(x)\zeta_1$. The rational function $c(x)$ can be expressed as $f_i/g_i$ for $i = 1, 2, 3$ and, further, we can assume that $c(x)$ is defined and nonzero for $x \notin \mathbb{T}^\Delta$.

*Proof.* If $x \in \mathcal{V}(J) \setminus \mathbb{T}^\Delta$ then we can assume without loss of generality that $f_1(x) \neq 0$ since the indeterminacy locus of $\zeta_1$ is contained in $\mathbb{T}^\Delta$. Suppose that $g_1(x) = 0$. Then since $g_1f_2(x) = g_2f_1(x)$, then $g_2(x) = 0$. Now since $g_1f_3(x) = g_3f_1(x)$, then $g_3(x) = 0$. This means that $x$ is in the indeterminacy locus of $\zeta_2$ and hence does not lie in the torus $\mathbb{T}^\Delta$. This is a contradiction, so if $f_1(x) \neq 0$, then $g_1(x) \neq 0$. Therefore, since
we assume in this section that $x \notin \mathbb{T}^\Delta$ throughout, we can assume without loss of generality that $c(x) = \frac{\ell}{g}$ and this quantity is defined as well as non-zero.

Let

$$P_{ev} = \prod_{i=1}^{k} x_{2i} \quad \text{and} \quad P_{od} = \prod_{i=1}^{k} x_{2i-1}$$

be the product of all even and odd-indexed variables, respectively.

**Lemma III.23.** The polynomial $P_{od} - P_{ev}$ vanishes on $\mathcal{W}$.

**Proof.** This follows immediately by evaluating at Wachspress coordinates.

**Conjecture III.24.** The polynomial $P_{od} - P_{ev}$ lies in the ideal $\hat{I}$.

We redefine $J$ to be the ideal generated by Wachspress quadratics, Wachspress cubics, the $d_i$, the three minors of $M_{fg}$, and $P_{od} - P_{ev}$. We now aim to show that for $x \in \mathcal{V}(J) \setminus \mathbb{T}^\Delta$ that $\beta(\zeta_m(x)) = x$ for $m = 1, \ldots, N$. First note it suffices to set $m = 1$ and show that $b_j(\zeta_1(x)) = x_j$ for any $j = 1, \ldots, N$. We wish to evaluate the expression

$$b_j(\zeta_1(x)) = \prod_{i \neq j-1,j} \ell_i(\zeta_1).$$

The next lemma shows us how to evaluate each factor $\ell_i(\zeta_1)$ in this product.

**Lemma III.25.** If $i \geq 2$ is even, then

$$\ell_i(\zeta_1(x)) = \frac{x_3x_5 \cdots x_{i-1}}{x_2x_4 \cdots x_{i-2}} M_{i-1,i},$$

and if $i \geq 2$ is odd, then

$$\ell_i(\zeta_1(x)) = \frac{1}{c(x)} \frac{x_4x_6 \cdots x_{i-1}}{x_3x_5 \cdots x_{i-2}} M_{i-1,i}.$$

Finally, for $i = 1$,

$$n_1 \cdot \zeta_1(x) = \frac{1}{c(x)} \frac{x_1x_3 \cdots x_{N-1}}{x_2} = \frac{1}{c(x)} \frac{x_2x_4 \cdots x_N}{x_2} = \frac{1}{c(x)} \frac{x_4x_6 \cdots x_N}{x_2}.$$
Proof. This follows by applying Lemma III.20 and Remark III.17.

Example III.26. Let $k = 3$. We work with the map $\zeta_1$, expressing each $\zeta_i$ in terms of $\zeta_1$.

$$\zeta_1 = \frac{1}{c(x)} \zeta_2 \quad \zeta_1 = \frac{x_3}{x_2} \zeta_3 \quad \zeta_1 = \frac{1}{c(x)} \zeta_2 = \frac{x_4}{c(x)x_3} \zeta_4$$

$$\zeta_1 = \frac{x_3 x_2 x_4}{x_2 x_3} \zeta_5 \quad \zeta_1 = \frac{x_4 x_6}{c(x)x_3x_5} \zeta_6$$

Using the above computations and Remark III.17 we can calculate

$$\ell_2(\zeta_1) = M_{1,2} \quad \ell_3(\zeta_1) = \frac{1}{c(x)} M_{2,3}$$

$$\ell_4(\zeta_1) = \frac{x_4}{x_2} M_{3,4} \quad \ell_5(\zeta_1) = \frac{x_4}{c(x)x_3} M_{4,5}$$

$$\ell_6(\zeta_1) = \frac{x_3 x_4 x_6}{x_2 x_4} M_{5,6} \quad \ell_1(\zeta_1) = \frac{x_4 x_6}{c(x)x_3x_5} M_{6,1}.$$

Further, we can use this to compute the following:

$$b_1(\zeta_1) = \ell_2(\zeta_1) \ell_3(\zeta_1) \ell_4(\zeta_1) \ell_5(\zeta_1)$$

$$= (M_{1,2})(\frac{1}{c(x)} M_{2,3})(\frac{x_3}{x_2} M_{3,4})(\frac{x_4}{c(x)x_3} M_{4,5})$$

$$= \frac{1}{c(x)^2} M_{1,2} M_{2,3} M_{3,4} M_{4,5}$$

$$= \frac{1}{c(x)^2} \frac{x_4}{x_2} P_{ev} P_{od} P_{ev} P_{od}$$

$$= \frac{1}{c(x)^2} \frac{x_4}{x_2} \frac{x_3}{x_4} \frac{x_4}{x_5}$$

$$= \frac{1}{c(x)^2} \frac{x_3 x_4}{x_2 x_3 x_5} = \frac{1}{c(x)^2} \frac{x_3 x_4}{x_2 x_3 x_4}$$

$$= \frac{1}{c(x)^2} \frac{P^2}{x_2 x_3 x_4}, \text{ and}$$
\[ b_2(\zeta_1) = \ell_3(\zeta_1)\ell_4(\zeta_1)\ell_5(\zeta_1)\ell_6(\zeta_1) \]
\[ = \left( \frac{1}{c(x)} M_{2,3} \right) \left( \frac{x_3}{x_2} M_{3,4} \right) \left( \frac{x_4}{c(x)x_3} M_{4,5} \right) \left( \frac{x_5}{x_2x_4} M_{5,6} \right) \]
\[ = \frac{1}{c(x)^2} \frac{x_3x_5}{x_2^2} M_{2,3} M_{3,4} M_{4,5} M_{5,6} \]
\[ = \frac{1}{c(x)^2} \frac{x_3x_5}{x_2^2} P_{od} P_{ev} P_{od} P_{ev} \]
\[ = \frac{1}{c(x)^2} \frac{x_3x_5}{x_2^2} \frac{P_{od}^2}{P_{ev}^2} \]
\[ = \frac{1}{c(x)^2} \frac{x_2^2}{x_4 x_6} \]
\[ = \frac{1}{c(x)^2} \frac{x_2}{x_4} M_{1,2}. \]

Now we claim that
\[ \frac{1}{x_1} b_1(\zeta_1) = \frac{1}{x_2} b_2(\zeta_1). \] (3.9)

From our calculation above and after some canceling, Equation 3.9 reduces to
\[ \frac{1}{x_1 M_{6,1}} = \frac{1}{x_2 M_{1,2}}, \]
but this simply says that
\[ \frac{1}{P_{od}} = \frac{1}{P_{ev}} \Rightarrow P_{od} = P_{ev}, \]
which we know is true.

**Example III.27.** Let \( N = 8 \).

\[ n_1(\zeta_1) = \frac{1}{c(x)} \frac{x_2 x_3 x_8}{x_3 x_3 x_7} M_{8,1} \quad n_2(\zeta_1) = M_{1,2} \]
\[ n_3(\zeta_1) = \frac{1}{c(x)} M_{2,3} \quad n_4(\zeta_1) = \frac{x_1}{x_2} M_{3,4} \]
\[ n_5(\zeta_1) = \frac{1}{c(x)} \frac{x_4}{x_3} M_{4,5} \quad n_6(\zeta_1) = \frac{x_5}{x_2 x_4} M_{5,6} \]
\[ n_7(\zeta_1) = \frac{1}{c(x)} \frac{x_4 x_6}{x_3 x_5} M_{6,7} \quad n_8(\zeta_1) = \frac{x_5 x_7}{x_2 x_4 x_6} M_{7,8} \]
Similarly, we will have

\[ b_2(\zeta_1) = \frac{1}{c(x)^3} \frac{1}{x_2} \frac{P_{ev} P_{od} P_{ev} P_{od} P_{ev} P_{od} P_{od}}{x_3 x_4 x_5 x_6 x_7} = \frac{1}{c(x)^3} \frac{1}{x_2^3} \frac{P^3}{M_{8,1}} \]

Observe that we have,

\[ \frac{1}{x_1} b_1(\zeta_1) = \frac{1}{x_2} b_2(\zeta_2). \]  \hspace{2cm} (3.10)

After canceling, Equation 3.10 reduces to

\[ \frac{1}{x_1} \frac{1}{M_{8,1}} = \frac{1}{x_2} \frac{1}{M_{1,2}}, \]

and this reduces to

\[ P_{od} = P_{ev}, \]

which does hold.

The two examples computed above can easily be generalized to conclude the following.

**Lemma III.28.**

\[ b_i(\zeta_1) = \frac{1}{c(x)^{k-1}} \frac{p^{k-1}}{x_2^{k-1} M_{i-1,i}} \]

We have seen that \( b_i(\zeta_1) = x_i f(x)/c(x)^{k-1} \) for some monomial \( f(x) \). From this it
follows that $\beta(\zeta_1(x)) = f(x)/c(x)^{k-1} x$ and this is equal to $x$ in $\mathbb{P}^\Delta$ since $f(x)/c(x)^{k-1}$ is defined and does not vanish on the complement of the torus $T^\Delta$. 
CHAPTER IV

CONCLUSION

A. Intersection with a Coordinate Hyperplane

We have shown in the previous chapter that any point \( x \in V(J) \setminus T^\Delta \) lies on \( W \). To obtain our set-theoretic result, Theorem IV.4, we must prove that the intersection of any coordinate hyperplane \( V(x_i) \) with \( V(J) \) is contained in \( W \). This will allow us to conclude that \( V(J) = W \). The ideal \( \hat{I} \) is generated by the Wachspress quadratics and cubics and the ideal \( K \) is generated by the Wachspress cubics. In this chapter we first investigate how \( V(\hat{I}) \) intersects a coordinate hyperplane. This will allow us to conclude our main result Theorem IV.4.

Later in the chapter we use the expression of the Wachspress quadratics and cubics as scalar products and determinants to describe a set of syzygies among them. We conclude by looking at some examples of Wachspress varieties and the ideal we have found that cuts them out for small \( N \).

Lemma IV.1. For any \( i = 1, \ldots, N \) the ideal \( \langle x_i \rangle + K \) is the monomial ideal generated by \( x_i \) and all \( \Delta \)-monomials not involving \( x_i \).

Proof. It suffices to show this for \( i = 1 \). If \( x_1 = 0 \), then we have \( \Lambda_1 = x_2n_2 \) and \( \Lambda_N = x_Nn_1 \). From this it immediately follows that \( w_{135} = |n_{246}|x_2x_4x_6 \) and \( w_{N-4,N-2,N} = |n_{N-5,N-3,N-1}|x_{N-4}x_{N-2}x_N \). Using that \( x_1 = 0 \) and \( x_2x_4x_6 = x_{N-4}x_{N-2}x_N = 0 \), we can recursively show that each of the remaining Wachspress cubics \( w_{ijk} \) either reduce to zero or to a \( \Delta \)-monomial. In fact every \( \Delta \)-monomial not involving \( x_1 \) will occur in this way.

Lemma IV.2. The intersection of \( V(K) \) with the coordinate hyperplane \( V(x_i) \) is a union of three dimensional coordinate planes.
Proof. It suffices to let $i = 1$. The monomial ideal generated by $x_1$ and all $\Delta$-monomials not involving $x_1$ has primary decomposition consisting of all ideals of the form $(x_1, x_{i_1}, \ldots, x_{i_{N-5}})$ where each $\Delta$-monomial not involving $x_1$ involves a variable in the set $\{x_{i_1}, \ldots, x_{i_{N-5}}\}$. \hfill \square

Lemma IV.3. $V(\hat{I}) \cap V(x_i)$ is the union of $\deg W = (N^2 - 5N + 8)/2$ lines. It contains all edge images except $\hat{\ell}_{i-1}$ and $\hat{\ell}_i$. It also contains the $(N - 3)(N - 4)/2$ blown up base points $\hat{p}_{j,k}$ where neither $j$ nor $k$ is in $\{i - 1, i\}$.

Proof. We calculate the lines that are expected when $x_1 = 0$. We expect to get all edge images $\hat{\ell}_j$ except $\hat{\ell}_1$ and $\hat{\ell}_N$, so $N - 2$. We also expect to get all blow ups of the base points $p_{ij}$ that do not meet either $\ell_1$ or $\ell_N$. There are $N(N - 3)/2$ base points and $N - 3$ of them meet $\ell_1$ and another $N - 3$ meet $\ell_N$. So we expect to get

$$N(N - 3)/2 - 2(N - 3) = (N - 3)(N - 4)/2 \quad (4.1)$$

blown up base points. In total then we expect to get

$$N - 2 + (N - 3)(N - 4)/2 = (N^2 - 5N + 8)/2 \quad (4.2)$$

lines. The ideal generated by the Wachspress cubics and $x_i$ is the monomial ideal generated by $x_i$ and all $\Delta$-monomials not involving $x_i$. By Lemma IV.2 this ideal’s components are all ideals of the form $(x_{i_1}, \ldots, x_{i_{N-4}})$ where $x_1 = 1$ and at least one of the variables of $m$ is contained in $\{i_2, \ldots, i_{N-4}\}$ for each $\Delta$-monomial $m$ not involving $x_1$. Suppose we are on one such component $(x_{i_1}, \ldots, x_{i_{N-4}})$. The component cuts out a three dimensional coordinate plane. We impose Wachspress quadratics on this component. Only four variables are nonzero on each component so most quadratics reduce to zero since many of the vectors $\Lambda_i$ will vanish. The remaining ones will cut down the three plane to a blown-up line or edge image. \hfill \square
We now state our main result whose proof follows from the lemmas and theorems in this and previous chapters.

**Theorem IV.4.** The Wachspress variety $W$ is cut out set-theoretically by the ideal $J$.

Once Conjectures III.11, III.19, and III.24 are proven this result will be strengthened to the following.

**Conjecture IV.5.** The Wachspress variety $W$ is cut out set-theoretically by the ideal generated by Wachspress quadratics and cubics.

B. Syzygies and Betti Diagrams

Given the expressions of the Wachspress quadratics and cubics as determinants and scalar products, we can use basic vector calculus identities to easily construct syzygies among them. Let $a$, $b$, and $c$ be vectors in three space. From vector calculus we have the identities:

\[
| a b c | = a \cdot (b \times c) = (a \times b) \cdot c
\]
\[
b(a \cdot c) = a \times (b \times c) + c(a \cdot b).
\]

**Theorem IV.6.** The relation $\pi w_{i,j,k} = (\Lambda_i \times \Lambda_j)q_k - (\Lambda_i \times \Lambda_k)q_j + (\Lambda_j \times \Lambda_k)q_i$ of vectors of three linear forms holds for any $\Delta$-triple $(i,j,k)$. In each coordinate, this vector relation yields a syzygy among the Wachspress quadratics and cubics.
Proof. Using the vector identities above we can write

\[
\pi w_{i,j,k} = \pi(x)(\Lambda_i \cdot \Lambda_j \times \Lambda_k) \\
= \Lambda_i \times (\pi(x) \times (\Lambda_j \times \Lambda_k)) + (\Lambda_j \times \Lambda_k)\Lambda_i \cdot \pi(x) \\
= \Lambda_i \times [\Lambda_j(\pi(x) \cdot \Lambda_k) - \Lambda_k(\pi(x) \cdot \Lambda_j)] + (\Lambda_j \times \Lambda_k)q_i \\
= \Lambda_i \times [\Lambda_j q_k - \Lambda_k q_j] + (\Lambda_j \times \Lambda_k)q_i \\
= (\Lambda_i \times \Lambda_j)q_k - (\Lambda_i \times \Lambda_k)q_j + (\Lambda_j \times \Lambda_k)q_i.
\]

Think of both sides of the equation in the statement of the theorem as column vectors of three forms. The top row writes \( w_{i,j,k} \) times the first generator of the center \( C \) as a combination of the Wachspress quadrics \( q_i, q_j, \) and \( q_k \). The second row expresses \( w_{ijk} \) times the second generator similarly and so on. \( \square \)

Theorem IV.6 accounts for many of the syzygies in the first syzygy module of \( \hat{I} \). In fact there is computational evidence supporting the next conjecture.

**Conjecture IV.7.** The Koszul syzygies and those from Theorem IV.6 generate the first syzygy module of \( \hat{I} \).

It remains to investigate the higher syzygies. The crucial information about syzygies is displayed in Betti diagrams. Using the computer algebra system Singular [16], Betti diagrams for Wachspress varieties for \( n \)-gons with \( n \leq 12 \). Some examples of these diagrams are displayed in Table III. It follows from the Auslander-Buchsbaum Formula [17] that a variety is Cohen-Macaulay if the length of its minimal free resolution is equal to its codimension. Thus, the Betti diagrams in Table III are evidence for the following conjecture.

**Conjecture IV.8.** Wachspress varieties are Cohen-Macaulay.
Table III. Betti Diagrams for \( n = 5, 6, \) and 7, respectively

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C. Examples for Small \( N \)

We end this Chapter with a close look at Wachspress varieties for \( N \)-gons with \( N = 3, 4, 5, 6, \) and 7.

**Example IV.9.** Let \( N = 3 \). The Wachspress coordinates for a triangle are \( b_1 = \ell_2, \)
\( b_2 = \ell_3, \) and \( b_3 = \ell_1 \). Since the linear forms \( \ell_i \) cut out the edges of a triangle the
Wachspress map \( \beta = [\ell_2, \ell_3, \ell_1] \) is an automorphism of \( \mathbb{P}^2 \). Hence, \( W = \mathbb{P}^2 \) and the
center of projection \( C \) is empty.

**Example IV.10.** Let \( N = 4 \). The Wachspress coordinates for a quadrilateral are
\( b_1 = \ell_2\ell_3, \) \( b_2 = \ell_3\ell_4, \) \( b_3 = \ell_1\ell_4, \) and \( b_4 = \ell_1\ell_2 \). In this case \( W \) is cut out by one
Wachspress quadratic and, thus, is a quadric surface in \( \mathbb{P}^3 \). The center \( C \) is a point
lying on \( W \). The adjoint curve \( A \) is a line through the two base points and is contacted
to the center point by the Wachspress map \( \beta \).

**Example IV.11.** Let \( N = 5 \). The Wachspress coordinates for a pentagon are \( b_1 = \ell_2\ell_3\ell_4, \)
\( b_2 = \ell_3\ell_4\ell_5, \) \( b_3 = \ell_1\ell_4\ell_5, \) \( b_4 = \ell_1\ell_2\ell_5, \) and \( b_5 = \ell_1\ell_2\ell_3 \). There are five Wachspress
quadratics but only two are linearly independent, and there are no Wachspress cubics.
In this case \( W \) is cut out by two quadratics in \( \mathbb{P}^4 \). The center of projection \( C \) is a
line contained in \( W \). It is the image of the adjoint curve under \( \beta \) in this case. The
adjoint curve for the pentagon is the unique conic through the five base points and is mapped to the center line by $\beta$.

**Example IV.12.** Let $N = 6$. The Wachspress coordinates for a hexagon are each a product of four linear forms. This is the first case where $W$ is not cut out in degree two. The variety $W$ is cut out by three Wachspress quadratics and one Wachspress cubic which in this case happens to be a binomial. The Wachpress cubic has the simple form $\alpha_2\alpha_4\alpha_6 x_1 x_3 x_5 - \alpha_1\alpha_3\alpha_5 x_2 x_4 x_6$. This is the first case where $C$ is not contained in $W$. The center is a two-plane that meets $W$ in a degree three curve that is the image of cubic adjoint curve through the nine base points. The reducibility of the intersection $C \cap W$ in [10] stems from hexagon in that example having three parallel sets of edges. In this case three of the base points are collinear, each lying on the line at infinity, leading to a reducible adjoint curve through the nine base points.

**Example IV.13.** Let $N = 7$. The Wachspress coordinates for a heptagon are each a product of five linear forms. The Wachspress variety is cut out by four quadratics and four cubics in $\mathbb{P}^6$. The center $C$ is a three plane meeting $W$ in the image $\beta(A)$ of the adjoint curve interpolating the fourteen base points.
REFERENCES


