1. Consider the statement: For all real numbers $x$ and $y$, if $xy$ is rational, then $x$ is rational.
   (a) [3 points] Write the converse of this statement.
   
   For all real numbers $x$ and $y$, if $x$ is rational, then $xy$ is rational.

   (b) [3] Write the contrapositive of this statement.
   
   For all real numbers $x$ and $y$, if $x$ is irrational, then $xy$ is irrational.

   (c) [3] Write the negation of this statement.
   
   There exist real numbers $x$ and $y$ such that $xy$ is rational and $x$ is irrational.

   (d) [3] Which of the above four statements (the proposition, its converse (a), its contrapositive (b), its negation (c)) are true? (You need not justify your answer.)
   
   Its negation (c).

2. [10] Prove that for all integers $n$, $n$ is divisible by 3 if, and only if, $n^2$ is divisible by 3.

   First we will prove that if $n$ is divisible by 3, then $n^2$ is divisible by 3: Assume $n$ is divisible by 3, that is, $n = 3x$ for some integer $x$. Then $n^2 = (3x)^2 = 3(3x^2)$, which is divisible by 3.

   Next we will prove that if $n^2$ is divisible by 3, then $n$ is divisible by 3: Assume that $n^2$ is divisible by 3, that is, 3 divides the product $n^2 = n \cdot n$. Since 3 is prime, this implies that it divides one of the factors, $n$ or $n$, that is, it divides $n$.

3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = \begin{cases} 2n - 1, & \text{if } n \text{ is even} \\ 2n, & \text{if } n \text{ is odd} \end{cases}$
   (a) [3] Find $f(\{1, 2, 3, 4\})$.
   
   $\{2, 3, 6, 7\}$

   (b) [3] Find $f^{-1}(\{1, 2, 3, 4\})$.
   
   $\{1, 2\}$

   (c) [3] Is $f$ injective? Justify your answer.

   Yes. If $f(m) = f(n)$ for some integers $m, n$, then either both $m, n$ are even or both are odd, by the definition of $f$. If both $m, n$ are even, then we have $2m - 1 = 2n - 1$, if
which implies \( m = n \) (by adding 1 and dividing by 2 on both sides of the equation). If both \( m, n \) are odd, then we have \( 2m = 2n \), which implies \( m = n \).

(d) [3] Is \( f \) surjective? Justify your answer.

No. For example, the number 1 is not in the image of \( f \): If it were, then we would have \( 2n - 1 = 1 \) since odd numbers are images of even numbers and vice versa. Therefore \( 2n = 2 \), so \( n = 1 \), but \( f(1) = 2 \), a contradiction.

4. Let \( A \) and \( B \) be sets and let \( Y \) be a subset of \( B \).

(a) [5] Let \( f : A \to B \) be a surjective function. Prove that \( Y = f(f^{-1}(Y)) \).

Let \( y \in Y \). Since \( f \) is surjective, there is an \( x \in A \) such that \( y = f(x) \). Therefore \( x \in f^{-1}(Y) \) and \( y \in f(f^{-1}(Y)) \). It follows that \( Y \subseteq f(f^{-1}(Y)) \).

Let \( y \in f(f^{-1}(Y)) \). Then \( y = f(x) \) for some \( x \in f^{-1}(Y) \). Since \( x \in f^{-1}(Y) \), we have that \( f(x) \in Y \). That is, \( y \in Y \). This shows that \( f(f^{-1}(Y)) \subseteq Y \).

We have proven that \( Y = f(f^{-1}(Y)) \).

(b) [5] Show that the assumption that \( f \) be surjective in part (a) is necessary, by giving an example of sets \( A, B \), a subset \( Y \) of \( B \), and a function \( f : A \to B \) for which \( Y \neq f(f^{-1}(Y)) \).

One example is: Let \( A = \mathbb{R}, B = \mathbb{R} \), and let \( f : \mathbb{R} \to \mathbb{R} \) be the function defined by \( f(x) = x^2 \). Let \( Y = [-1, 1] \). Then \( f^{-1}(Y) = [-1, 1] \) and \( f(f^{-1}(Y)) = [0, 1] \). So \( Y \neq f(f^{-1}(Y)) \).

5. [14] Prove by induction that for each positive integer \( n \),

\[
1 + 3 + 5 + \cdots + (2n - 1) = n^2.
\]

First check that the statement is true when \( n = 1 \): The left side of the equation is 1, and the right side of the equation is \( 1^2 = 1 \).

Next assume the statement is true when \( n = k \) for some positive integer \( k \), that is, assume that

\[
1 + 3 + 5 + \cdots + (2k - 1) = k^2.
\]

Then

\[
1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = k^2 + (2(k + 1) - 1) = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k + 1)^2,
\]

as desired.

6. (a) [5] Use the Euclidean algorithm to find \((91, 35)\).

7.
(b) [5] Find integers $x$ and $y$ such that $(91, 35) = 91x + 35y$.

$x = 2, y = -5$.

7. [12] Let $R$ be the relation on $\mathbb{Z}$ defined by $aRb$ if $a \leq b + 1$. Determine whether $R$ is reflexive, symmetric, or transitive. Justify your answer.

$R$ is reflexive: $a \leq a + 1$ for all $a \in \mathbb{Z}$, so $aRa$ for all $a \in \mathbb{Z}$.

$R$ is not symmetric: For example, if $a = 0$ and $b = 2$, then $aRb$ but it is not true that $bRa$.

$R$ is not transitive: For example, if $a = 2, b = 1, c = 0$, then $aRb$ and $bRc$, but it is not true that $aRc$.

8. [10] Find the least positive integer $x$ that satisfies the congruence

$4x \equiv 32 \pmod{9}$.

Note that $4 \cdot 7 = 28$, which is equivalent to 1 modulo 9. Also, 32 is equivalent to 5 modulo 9. Multiplying both sides of the equation by 7, we then have that $x$ is equivalent to $7 \cdot 5 = 35 \equiv 8$, so $x = 8$ is the answer.

9. [10] Prove that the function $f(x) = \frac{1}{1 + x^2}$ defines a bijection from $\mathbb{R}^+$ (the set of positive real numbers) to the open interval $(0, 1)$.

First note that if $x \in \mathbb{R}^+$ then $f(x) \in (0, 1)$.

$f$ is injective: Let $x_1, x_2 \in \mathbb{R}^+$ such that $f(x_1) = f(x_2)$. That is, $\frac{1}{1 + x_1^2} = \frac{1}{1 + x_2^2}$, and we may solve this equation e.g. for $x_1$ to obtain $x_1 = x_2$. (Note that at one point in the solution process, we need to take the square root of both sides of the equation $x_1^2 = x_2^2$. In general, this would result in two solutions, $x_1 = \pm x_2$. However, we have assumed that both $x_1$ and $x_2$ are positive, so we only take the positive solution.)

$f$ is surjective: Let $y \in (0, 1)$. We wish to show that there is an $x \in \mathbb{R}^+$ for which $y = f(x)$, that is, $y = \frac{1}{1 + x^2}$. Solving for $x$, we obtain $x = \sqrt{\frac{1-y}{y}}$, which is an element of $\mathbb{R}^+$, as desired. So $y = f(x)$ for this value of $x$. (Note that in solving the equation, at one step we need to take the square root of both sides, and so we need to consider the negative square root as well as the positive square root. However, we were only interested in the positive square root, since we wanted to find a value in $\mathbb{R}^+$.)