Math 608, Homework 12

1. Exercise 6.41. You may assume that the space is $\sigma$-finite.

2. Prove the Marcinkiewicz interpolation theorem in the case $p_0 = q_0$, $p_1 = q_1 = \infty$, by adapting the proof given in class, or filling in the details in Case I of the proof in the textbook.

3. Exercise 6.43.

4. Exercise 6.44.

5. (Compare with Exercise 6.45, which however has a misprint.) Use Theorem 6.36 to prove the following version of Sobolev’s inequality.

Let $0 < \alpha < n$, and define the operator $T_\alpha$ on functions on $\mathbb{R}^n$ by

$$T_\alpha f(x) = f * |x|^{-\alpha} = \int |x - y|^{-\alpha} f(y) \, dy.$$  

Then for

$$\frac{1}{p} + \frac{\alpha}{n} = \frac{1}{r} + 1$$

$T_\alpha$ is of weak type $(p, r)$ (in particular, it is of weak type $(1, n/\alpha)$), and it is of strong type $(p, r)$ if $p > 1$.

6. Prove the theorem from Exercise 5 directly, by filling in the details in the following argument.

   (a) Denote $k(x) = |x|^{-\alpha}$. Write $k = k_0 + k_\infty$, where $k_0 = k\chi_{|x| \leq A}$ and $k_\infty = k\chi_{|x| > A}$, for $A$ to be chosen. Note that $k_0$ has a singularity only at 0, $k_\infty$ only at infinity. Use the results from Section 2.7 to compute $\|k_0\|_1$ and $\|k_\infty\|_{p'}$, in particular concluding that these norms are finite. It follows that for $f \in L^p$,

$$f * k = f * k_0 + f * k_\infty \in L^p * L^1 + L^p * L^{p'} \subset L^p + L^\infty$$

is well defined.

(b) Given $t > 0$, choose $A$ so that $\|f * k_\infty\|_\infty \leq t/2$, and conclude that $\lambda_{f * k_\infty}(t/2) = 0$.

(c) For this $A$, show that

$$\lambda_{T_\alpha f}(t) \leq Ct^{-r}$$

and so $T_\alpha f \in L^r_w$.

(d) Complete the argument.

Quiz 12: Exercise 6.42.