consider finding a particular solution to nonhom. D.E.
\[ y'' + p(x)y' + q(x)y = g(x) \]  \( \text{(1)} \)

Recall: If \( y_1 \) and \( y_2 \) form a fundamental set of solutions to the hom. D.E.
\[ y'' + p(x)y' + q(x)y = 0 \]  \( \text{(2)} \)
then the general solution is
\[ y(x) = c_1 y_1(x) + c_2 y_2(x) \]
we look for a particular solution to the nonhom. D.E.
\( u(x) \) of the form
\[ u(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \]
for some unknown functions \( v_1(x) \) and \( v_2(x) \).
we have
\[ y' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2' \]
since we do not wish involve \( v_1'' \) and \( v_2'' \), we assume
\[ v_1' y_1 + v_2' y_2 = 0 \]  \( \text{(3)} \)
then
\[ y' = v_1 y_1' + v_2 y_2' \]
\[ y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2' + v_2 y_2' \]
plugging into the D.E. \( u \), we obtain
v_1'y_1' + v_2'y_2' + v_2'y_2' + p(x)[v_1'y_1' + v_2'y_2'] + p(x)[v_1'y_1' + v_2'y_2'] = g(x)

\Rightarrow \quad v_1'y_1' + v_2'y_2' = g(x) \quad \Rightarrow \quad v_1' = \frac{-p(x)y_2'}{W[y_1, y_2]} , \quad W[y_1, y_2] = y_1'y_2' - y_1'y_2.

v_1'y_1' + v_2'y_2' = 0 \quad v_2' = \frac{-p(x)y_1'}{W[y_1, y_2]}

Thus

\[ v_1(x) = \int \frac{-p(x)y_2'}{W[y_1, y_2]} \, dx, \quad v_2(x) = \int \frac{p(x)y_1'}{W[y_1, y_2]} \, dx. \]

\[ Y(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \]

is a particular solution to the nonhom. D.E. \( u_0 \) and the general solution to the nonhom. D.E. \( u \) is

\[ Y(x) = C_1y_1(x) + C_2y_2(x) + Y(x). \]

Ex. 3) Find a particular solution \( Y(x) \) of the nonhom. D.E.

\[ y'' + y = \tan x \quad -\frac{\pi}{2} < x < \frac{\pi}{2}; \]

4) Find a solution of the D.E. s.t. \( Y(\frac{\pi}{2}) = 1, \ Y'(0) = 1 \).

Sol. Char. eq. \( r^2 + 1 = 0, \ r = \pm i \), \( y_1(x) = \cos x, \ y_2(x) = \sin x \).

\[ W[y_1, y_2] = \cos^2 x - (-\sin^2 x) = 1. \]

\[ v_1 = -\int \frac{p(x)y_2}{W[y_1, y_2]} \, dx = \int \tan x \cdot 5\sin x \, dx = 5\int \frac{\sin^2 x}{\cos x} \, dx = \int \frac{-\cos^2 x}{\cos x} \, dx \]

\[ = \int \cos x \, dx = -\sec x \, dx = \sin x - \ln |\sec x + \tan x| \]
\( v_2(x) = \int_{y_2(x)}^{y_1(x)} \frac{q(x, y_2)}{W(y_1, y_2)} \, dx = \int \tan x \cdot \cos x \, dx = \int \sin x \, dx = -\cos x. \)

\( y(x) = v_1(x) y_1(x) + v_2(x) y_2(x) = (\sin x - \ln |\sec x + \tan x|) \cos x - \cos x \cdot \sin x = -\cos x \ln |\sec x + \tan x|. \)

b) The general solution is

\[ y(x) = C_1 \cos x + C_2 \sin x - \cos x \ln |\sec x + \tan x|. \]

I.C. \( y(0) = C_1 \cdot \cos x \left( \frac{\cos x}{\sec x + \tan x} \right) \bigg|_{x=0} = C_2 = 2. \)

Thus

\[ y(x) = \cos x + 2 \sin x - \cos x \ln |\sec x + \tan x|. \]

Ex: Verify that \( y_1(x) = x \) and \( y_2(x) = -\frac{1}{x} \) are solutions to

\[ x^2 y'' + xy' - y = 0. \]

and then find the general solution to

\[ x^2 y'' + xy' - y = x \ln x, \quad x > 0. \]

Sol: a) Verify directly. Rewrite \( y'' + \frac{y'}{x} - \frac{1}{x^2} y = \frac{\ln x}{x}. \)

\[ b) W[y_1, y_2] = \left| \begin{array}{cc} y_1 & y_2 \\ y_1' & y_2' \end{array} \right| = \frac{x}{\ln x} + \frac{1}{x^2} x = \frac{1}{x^2} \ln x \cdot x. \]

\[ v_1(x) = \int \frac{q(x, y_1)}{W(y_1, y_2)} \, dx = \int \frac{\ln x}{x} \, dx = \frac{1}{4} \ln x^2, \]

\[ v_2(x) = \int \frac{q(x, y_2)}{W(y_1, y_2)} \, dx = \int \frac{1}{x^2} \ln x \cdot x \, dx = \frac{1}{2} \int x \ln x \, dx = \frac{1}{4} x^2 (\ln x - \frac{1}{2}). \]
The general solution is

\[ y(x) = C_1 y_1(x) + C_2 y_2(x) + y_3(x) + y_4(x) + y_5(x) + y_6(x) \]

\[ = C_1 x + C_2 \frac{1}{x} + x^2 \ln x - \frac{x^2}{4} (\ln x - \frac{1}{2}) \cdot \frac{1}{x} \]

\[ = C_1 x + C_2 \frac{1}{x} + \frac{1}{x} \ln x^2 - \frac{1}{2} x^2 \ln x. \]

In general \( y'' = f(x, y, y') \)

1) D.E. with \( y \) missing: \( y'' = f(x, y') \). Set \( v = y' \)
   then \( v' = y'' \). We have \( v' = f(x, v) \) 1st-order D.E.
   solve it for \( v \), then \( y = \int v \, dx \).

2) D.E. with \( x \) missing: \( y'' = f(y, y') \) \( \Rightarrow \) set \( v = y' \)
   then \( v' = f(y, v) \). \( v' = \frac{dv}{dx} = \frac{dv}{dy} \frac{dy}{dx} = v \frac{dv}{dy} \).
   Thus \( v \frac{dv}{dy} = f(y, v) \). 1st-order D.E.
   solve it for \( v = v(y') \). Then solve \( y = y(x) \) from \( \frac{dy}{dx} = v(y') \).

Ex: Solve \( xy'' + y' = 1, x > 0 \). Set \( v = y' \). \( x v' + v = 1 \).

\[ \frac{v'}{1-v} = \frac{1}{x} \Rightarrow \int \frac{v'}{1-v} \, dv = \int \frac{1}{x} \, dx \Rightarrow -\ln(1-v) = \ln x + C. \]

\[ 1-v = \frac{C}{x} \Rightarrow v = 1 - \frac{C}{x} \]

\[ y(x) = \int \frac{d(x C)}{x} dx + C_2 = \int \left(1 - \frac{C}{x} \right) dx + C_2 \]

\[ = x - C_1 \ln x + C_2. \]