In general \( \frac{dy}{dt} = f(t, y) \).

Autonomous if \( \frac{dy}{dt} = f(y) \), always separable
solvable if \( \int \frac{dy}{f(y)} = t + C \) is solvable.

Population Models.

Case 1: \( f(y) = ay + b \)

Exponential growth: \( \frac{dy}{dt} = ry \), \( r > 0 \)
I.C. \( y_0 = y_0 \Rightarrow y(t) = y_0 e^{rt} \) exponential growth if \( y_0 > 0 \).
It is true for at least some time period, cannot continue exponentially due to limitation in space, food supply etc. need to modify the model.

Logistic growth: \( \frac{dy}{dt} = g(y)y \) where

\[ g(y) \begin{cases} > 0 & \text{if } y \text{ is small (grows)} \\ < 0 & \text{if } y \text{ is large (decays)} \end{cases} \]

One of the choices is \( g(y) = r - ay \), \( r > 0, a > 0 \)
such \( g(y) \) satisfies the property \((x)\).
We have
\[
\frac{dy}{dt} = (r - ay) y = r(1 - \frac{y}{K}) y
\]
where \( K = \frac{R}{a}, \ r = \text{intrinsic growth rate.} \)

Def. For \( \frac{dy}{dt} = fy(y) \), an equilibrium solution (critical point) is a scalar \( y \) s.t. \( fy(y) = 0 \).

It is clear that for \( fy(y) = r(1 - \frac{y}{K}) y, \ \phi_1 = 0, \ \phi_2 = K \) are two equilibrium solutions.

Def. An equilibrium solution is stable, if it can sustain small perturbation in I.C. otherwise the equilibrium solution is unstable.

Ex. \( \frac{dy}{dt} = (1 - \frac{y}{3}) y \).

1) determine its critical pts. \( \phi_1 = 0, \ \phi_2 = 3 \),
2) solve \( f(y) = 0 = 1 - \frac{2}{3} y \)
   get \( y = \frac{3}{2}, \ f\left(\frac{3}{2}\right) = \frac{3}{4} \)
3) plot \( f(y) \) against \( y \).
   determine the sign of \( f(y) \) near each critical pt.
   draw \( f(y) \) at each critical pt if \( f(y_*) \) is \( >0 \) \( y_* \) is stable, \( f(y_*) \) is \( <0 \) \( y_* \) is unstable.
For \( f'(y) = r(1 - \frac{y}{K})y \), \( r > 0 \).
\[ f'(y) = r(1 - \frac{y}{K}) = 0. \]
\[ y = \frac{K}{2}, \quad f\left(\frac{K}{2}\right) = r \frac{K}{4}. \quad \text{stable} \]

Find more information on \( y = y(t) \).
\[ \frac{d^2y}{dt^2} > 0 \quad \text{up} \]
\[ \frac{d^2y}{dt^2} < 0 \quad \text{down} \]
\[ \frac{d^2y}{dt^2} = 0 \quad \text{inflection point where} \]
\[ y \text{ changes its concavity} \]
Since \[ \frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{df}{dt} = f'(y) \frac{dy}{dt} = f'(y)f(y). \]

for \( f(y) = r(1 - \frac{y}{K})y \), \( \frac{d^2y}{dt^2} = r^2(1 - \frac{2y}{K})(1 - \frac{y}{K})y \)

To find \( y = y(t) \), from \( \frac{dy}{dt} = r(1 - \frac{y}{K})y \), separable
\[ \Rightarrow \frac{dy}{y(1 - \frac{y}{K})} = r dt. \quad \text{Use partial fraction} \Rightarrow \]
\[ \left(\frac{1}{y} + \frac{1}{K - y}\right)dy = r dt \]
\[ \Rightarrow \ln|y| - \ln|1 - \frac{y}{K}| = rt + c. \]
\[ \Rightarrow \ln\left|\frac{y}{K - y}\right| = rt + c \] or \( \frac{y}{K - y} = Ce^{rt} \)
\[ y(t + \frac{c}{Ke^{rt}}) = Ce^{rt} \Rightarrow y(t) = \frac{Ce^{rt}}{1 + \frac{c}{Ke^{rt}}} = \frac{cK}{1 + \frac{c}{Ke^{rt}} + c}. \]

we have \( \lim_{t \to \infty} y(t) = K \). \( y(t) \) is.
To find \( C \) by I.C.
\[ y(0) = y_0, \quad \Rightarrow \frac{ck}{K + c} \Rightarrow C = \frac{y_0K}{K - y_0}. \]
Conclusion: $\Phi_1 = 0$ is unstable,
$\Phi_2 = K$ is asymptotically stable and 
\[ \lim_{t \to \infty} y(t) = K, \quad y(0) = \varepsilon > 0. \]

A critical threshold, consider \( \frac{dy}{dt} = -r(1 - \frac{y}{T})y \)
when $y_0 > T$, there is $t^* > 0$ s.t. $y_0 + (T - y_0)e^{-rt^*} = 0$
$\Rightarrow y(t^*) = \infty$, blow up. Indeed we have
\[ y(t) = \frac{y_0 T}{y_0 + (T - y_0)e^{-rt}}, \quad t^* = \frac{1}{r} \ln \frac{y_0}{y_0 - T}. \]
$T$ = critical threshold.

Logistic growth with a threshold
\[ \frac{dy}{dt} = f(y) = -r(1 - \frac{y}{T})(1 - \frac{y}{K})y, \quad r > 0, \quad 0 < T < K, \]
$\Phi_1 = 0$, $\Phi_2 = T$, $\Phi_3 = K$, three critical pts.

$\Phi_1 = 0$ stable, $\Phi_2 = T$ unstable, $\Phi_3 = K$ stable.