§ 10.2. Green's Theorem

Let $D$ be a closed, bounded region in $\mathbb{R}^2$, whose boundary $C = \partial D$ consists of finitely many simple closed curves that orient the curve $C$ s.t. $D$ is on the left. Let $F(x, y) = M(x, y) \hat{i} + N(x, y) \hat{j}$ be a $C'$ vector field. Then

$$\oint_C M \, dx + N \, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy.$$ 

Ex. Let $F = x^2 \hat{i} + y^2 \hat{j}$, $D$ bounded by $y = x^2$.

LHS = $\int_0^\infty (x^2 + x^2 e^{2t}) \, dt + \int_0^1 (x^2 + x^2) \, dt$

$= \frac{1}{4} + 2 \frac{1}{6} - \frac{2}{3} = -\frac{1}{12}$

RHS = $\iint_D (1 - x) \, dx \, dy = \int_0^1 \left( \int_y^{\sqrt{y}} x^2 \, dx \right) \, dy = \int_0^1 \frac{x^3}{2} \bigg|_y^{\sqrt{y}} \, dy$

$= \int_0^1 \left( \frac{y}{2} + \frac{y^3}{2} \right) \, dy = -\frac{1}{4} + \frac{1}{6} = -\frac{1}{12}$.

For $F = -y \hat{i} + x \hat{j}$, $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 + 1 = 2$. Then

$$\oint_C M \, dx + N \, dy = \iint_D \frac{2}{x} \, dx \, dy = 2 \iint_D \, dx \, dy = 2 \text{ area of } D.$$
Ex: Find the area of the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

\[ x = a \cos t, \quad y = b \sin t, \quad 0 \leq t \leq 2\pi \]

Area: \( \frac{1}{2} \int_C (\mathbf{F} \cdot d \mathbf{r}) \) where \( \mathbf{F} = (x, y) \)

\[ = \frac{1}{2} \int_0^{2\pi} (\mathbf{F} \cdot d \mathbf{r}) = \frac{1}{2} \int_0^{2\pi} ab \, dt = ab \pi. \]

Outer boundary counterclockwise

Inner boundary clockwise

\( n \) = the outward unit normal vector.

Let \( D \) be a region bounded by \( C = \partial D \) s.t. Green's theorem applies. Let \( n \) be the outward unit normal vector at \( C = \partial D \), and \( \mathbf{F}(x, y) = M(x, y)i + N(x, y)j \) be \( C \) vector field on \( D \). Then

\[ \mathbf{F} \cdot n = \text{particles crossing} \quad \text{C out} = \text{Flux} \]

\( \mathbf{F} \cdot n = \text{rate of particles leaving a point} \)

\[ \oint_C (F \cdot n) \, ds = \iint_D \nabla \cdot F \, dA \quad (\text{divergence theorem}) \]

Total particles crossing \( C \) out = total particles left \( \partial D \).
when $C : x(t) = (x(t), y(t))$.

$T(t) = (x'(t), y'(t)) \perp \mathbf{N}(t) = (-y'(t), x'(t))$, $\mathbf{n}(t) = \frac{\mathbf{N}(t)}{||\mathbf{N}(t)||}$.

Path Independence.

A vector field $\mathbf{F}$ is said to have a path-independence line integral if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{s} = \int_{C_2} \mathbf{F} \cdot d\mathbf{s}$$

for any simple, piecewise $C^1$ curves with the same initial and terminal points.

**Theorem.** A vector field $\mathbf{F}$ has a path-independent line integral if and only if

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = 0$$

for any piecewise $C^1$ simple, closed curve.

**Proof.**

$$\oint_C \mathbf{F} \cdot d\mathbf{s} = \int_{C_1} \mathbf{F} \cdot d\mathbf{s} - \int_{C_2} \mathbf{F} \cdot d\mathbf{s} = 0.$$
THEM. Let \( F \) be a continuous vector field on a connected open region \( D \) of \( \mathbb{R}^n \). Then \( F = \nabla f \) if and only if \( F \) has a path-independent line integral over curves in \( D \). Moreover, if \( C \) is any piecewise \( C^1 \)-oriented curve in \( D \) with initial point \( A \) and terminal point \( B \), then
\[
\int_C F \cdot dS = f(B) - f(A), \quad (F = \nabla f).
\]

**Proof:** \( C: x(t), \ A = x(a), \ B = x(b) \)
\[
\int_C F \cdot dS = \int_a^b F(x(t)) \cdot x'(t) dt = \int_a^b \frac{d}{dt} f(x(t)) dt
\]
\[
= f(x(b)) - f(x(a)) = f(B) - f(A).
\]

**Ex.** \( F = M \hat{i} + N \hat{j} = x \hat{i} + y \hat{j} \). Note \( \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \).
\[
\int_C F \cdot dS = \int_0^1 M dx + N dy = \int_0^1 (M x'(t) + N y'(t)) dt
\]

1) \( x = t, \ y = t, \ 0 \leq t \leq 1, \int_C F \cdot dS = \int_0^1 (x + x) dt = 1; \)
2) \( x = t, \ y = t^2, \ 0 \leq t \leq 1, \int_C F \cdot dS = \int_0^1 (t + t^2) dt = \frac{1}{2} + 
frac{2}{4} = 1; \)
3) \( x = 0, \ y = t, \ 0 \leq t \leq 1, \int_C F \cdot dS = \int_0^1 (0 + 0) dt + \int_0^1 (t dt + 0) dt = 1; \)
4) \( F = \frac{1}{2} (x^2 + y^2), \ F = \nabla f = (F_x, F_y) = (x, y), \)
\[
\int_C F \cdot dS = f(1, 1) - f(0, 0) = \frac{1}{2}(1 + 1) + 0 = 1.
\]
When $F = \nabla f$, $f$ is called a conservative vector field, scalar potential.

For given $F$,
1) How to know if $F$ is conservative?
2) Assume $F$ is conservative, how to find $f$ s.t. $F = \nabla f$?

Def. A region $D$ in $\mathbb{R}^2$ is simply connected if any simple closed curve in $D$ can be shrunk to a point.

Yes: $\bigcirc$  no: $\bigcirc$

Thm: Let $F = M \hat{i} + N \hat{j}$ be a $C^1$ vector field in a simply connected region $D$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then $F = \nabla f$ for some $f$ if and only if $\nabla \times F = 0$ in $D$.

Remark: $\nabla \times F = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & N \\ D & D & D \end{vmatrix} = \frac{\partial N}{\partial z} i - \frac{\partial M}{\partial z} j + (\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}) k = 0$

$\Rightarrow N = N(x, y), M = M(x, y)$ and $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$.

Ex: Let $F = x^2 y \hat{i} - 2xy \hat{j}$
then $\frac{\partial N}{\partial x} = -2y + \frac{\partial M}{\partial y} = x^2 \Rightarrow$ not conservative.
Ex. \( F = (2x + z + 2y) \hat{i} + (x^2 - 2x \sin 2y) \hat{j} \). Check. \( \frac{\partial M}{\partial y} = 2x - 25 \sin 2y = \frac{\partial N}{\partial x} = 2x - 25 \sin 2y \)

\[ \Rightarrow F \text{ is conservative. How to find } f \text{ s.t. } F = \nabla f ? \]

\[ F = M \hat{i} + N \hat{j} = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right). \]

\[ M = \frac{\partial f}{\partial x} \Rightarrow f = \int M \, dx + \alpha(y) = x^2y + x \cos 2y + \alpha(y) \]

\[ N = \frac{\partial f}{\partial y} \Rightarrow f = \int N \, dy + \beta(x) = x^2y + x \cos 2y + \beta(x) \]

\[ f = f \Rightarrow \alpha(y) = \beta(x) = 0 \Rightarrow f = x^2y + x \cos 2y. \]

**THM.** If \( D \) is simply connected domain, then \( F = \nabla f \) in \( D \) if and only if \( \nabla \times F = 0 \)

Ex. \( F = (e^x \sin y - y^2) \hat{i} + (e^x \cos y - x^2) \hat{j} + (z - xy) \hat{k} \)

Check \( \nabla \times F = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \end{vmatrix} = 0. \)

\[ \Rightarrow F = M \hat{i} + N \hat{j} + PK = \nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right). \text{ To find } f. \]

\[ f = \int M \, dx + \alpha(y, z) = \int (e^x \sin y - y^2) \, dx + \alpha(y, z) = e^x \sin y - xy^2 + \alpha(y, z) \]

\[ f = \int N \, dy + \beta(x, z) = \int (e^x \cos y - x^2) \, dy + \beta(x, z) = e^x \cos y - x^2y + \beta(x, z) \]

\[ f = \int P \, dz + \gamma(x, y) = \int (z - xy) \, dz + \gamma(x, y) = z^2 - xyz + \gamma(x, y) \]

\[ f = f = f \Rightarrow \alpha(y, z) = \beta(x, z) = 0, \gamma(x, y) = e^x \sin y \]

\[ \Rightarrow f = e^x \sin y - xy^2 + z^2 \, (c) \]
Next to compute $\int \mathbf{F} \cdot d\mathbf{S}$ along a curve from $(0, 0, 0)$ to $(1, \frac{\pi}{2}, 2)$, we have

$$\int \mathbf{F} \cdot d\mathbf{S} = f(1, \frac{\pi}{2}, 2) - f(0, 0, 0) = e^{-1} - 1 \cdot \frac{\pi}{2} \cdot 2 + \frac{4}{2} = e^{-1} + 2.$$