Linear Algebra - theory and Methods to solve linear systems, is the most useful math tool. 75% of math problems in science & engineering involves solving a linear system at a stage.

What is a linear system? A system of linear equations. A linear equation in n unknowns is of the form

\[ a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \]

where the coefficients \( a_1, \ldots, a_n \) and the right-hand side \( b \) are given real \( \#s \), and \( x_1, \ldots, x_n \) are \( n \) unknowns to be found.

A system of \( m \) linear equations in \( n \) unknowns is of the form

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
    \vdots & \quad \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

Where the coefficients \( a_{ij} \)'s and RHS \( b_i \)'s are given real \( \#s \), \( x_1, \ldots, x_n \) are \( n \) unknowns to be found.
Ex: 2x2 system; 2x3 system; 3x2 system

e) \begin{align*}
    x_1 + 2x_2 &= 5 \\
    2x_1 + 3x_2 &= 8
\end{align*}

(b) \begin{align*}
    x_1 - x_2 + x_3 &= 2 \\
    2x_1 + x_2 - x_3 &= 4
\end{align*}

(c) \begin{align*}
    x_1 + x_2 &= 2 \\
    x_1 - x_2 &= 1 \\
    x_1 &= 4
\end{align*}

Def: A solution to an mxn system is an ordered n-tuple \((x_1, \ldots, x_n)\) that satisfies all m equations.

Ex: \((x_1, x_2) = (1, 2)\) is a solution to (a); 
\((x_1, x_2, x_3) = (2, 0, 0)\) is a solution to (b). In fact, for any \(x\), \((x_1, x_2, x_3) = (2, x, x)\) is a solution to (b).

As for (c), the 3rd equation \(x_1 = 4\) leads to \(4 + x_2 = 2\) 
\(4 - x_2 = 1\)

\(\Rightarrow x_2 = -2\) \(\Rightarrow\) no solution.

\(\Rightarrow x_2 = 3\) \(\Rightarrow\) no solution.

* A linear system may have \{\)
  solutions (consistent) \{\) many solutions.
  \{\)
  solutions (Inconsistent) \{\) no solution.

(a) and (b) are consistent, (c) is inconsistent.

* The set of all solutions to a linear system is the solution set.
* By solving a linear system, we mean finding the solution set.

How to solve a linear system? How to find the solution set?
2x2 Systems. A 2x2 system is of the form
\[ \begin{align*}
    a_{11} x_1 + a_{12} x_2 &= b_1 \\
    a_{21} x_1 + a_{22} x_2 &= b_2
\end{align*} \]

In a 2D-plane, an equation \( a x_1 + b x_2 = c \) represents a straight line. So a 2x2 system represents two lines in a 2D-plane. There are totally 3 cases:

- Intersect
- Parallel
- Overlap

![Graphs showing consistent, inconsistent, and infinitely many solutions cases.]

A solution to a 2x2 system is the intersection points of the two lines.

A linear equation in 3 variables \( a_{11} x_1 + a_{22} x_2 + a_{33} x_3 = b \) represents a plane in a 3D-space.

Consider a 2x3 system:
\[ \begin{align*}
    a_{11} x_1 + a_{22} x_2 + a_{33} x_3 &= b_1 \\
    a_{21} x_1 + a_{22} x_2 + a_{33} x_3 &= b_2
\end{align*} \]

Totally 3 cases:

1) Two planes intersect (along a line) infinitely many solutions.
2) Two planes are parallel (no solution) inconsistent.
3) Two planes are the same (consistent).
* An mxn system may have no solution (inconsistent).

Solution(s) (consistent) only one infinitely many.

Def: Two systems are equivalent if they have exactly the same solution set.

Remark: Two equivalent systems must have the same # of variables.

Three elementary operations that will not change the solution set:
1) Interchange the order of two equations;
2) Multiply a nonzero # to both sides of an equation;
3) Add a multiple of an equation to another equation.

Two systems are equivalent if one system can be obtained by performing a sequence of three ele operations to another system.

Two systems are equivalent if one system can be obtained by first using those three ele operations and converting it into a simpler and equivalent system, then solve the simpler system.
Ex. (a): \[3x_1 + 2x_2 - x_3 = -2 \quad \text{and} \quad 3x_1 + 2x_2 - x_3 = -2\]

are equivalent, but (b) is much easier to solve for \( x_2 = 3 \) \( \Rightarrow x_1 = -2 \)

So to solve (a), we may convert (a) into (b) by using 3 ele operations: add (1) to (2) \( \Rightarrow x_2 = 3 \) \( \text{i.e., (b)} \).

n xn systems have many special properties.

Def. An nxn system is in a triangular form if in the k-th equation, the first k-1 coefficients are all zero, i.e., \( a_{ij} = 0 \) if \( i > j \). 

\[a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \quad 0 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \quad \vdots \]

\[x_n \Rightarrow x_{n-1}, \ldots \Rightarrow x_2 \Rightarrow x_1. \quad \text{back substitution.}\]

Ex. \[3x_1 + 2x_2 + x_3 = 1\]

\[x_2 - x_3 = 2 \quad \text{is in triangular form}\]

\[2x_3 = 4\]

Use back substitution: \( x_3 = 2, \Rightarrow x_2 - 2 = 0 \Rightarrow x_2 = 4, \Rightarrow 3x_1 + 8 + 2 = 10 \Rightarrow x_1 = -3. \text{ Thus } (x_1, x_2, x_3) = (-3, 4, 2) \text{ is the solution.}\]

* Any nxn system can be converted into a triangular form by using 3 ele operations.
Ex. \( x_1 + 2x_2 + x_3 = 3 \) \(-3\) \( x_1 + 2x_2 + x_3 = 3 \)
\( 3x_1 - x_2 - 3x_3 = -1 \) \(-2\) \( -7x_2 - 6x_3 = -10 \)
\( 2x_1 + 3x_2 + x_3 = 4 \)

\((2) \iff (3)\)

\( x_1 + 2x_3 + x_3 = 3 \)
\(-7 \iff (3) \Rightarrow x_1 + 2x_3 + x_3 = 3 \)

\(-x_2 - x_3 = -2 \Rightarrow -x_2 - x_3 = -2 \)
\(-7x_2 - 6x_3 = -10 \Rightarrow -7x_2 - 6x_3 = -10 \)

\( x_8 = 4 \)
\(-x_2 - 4 = -2 \Rightarrow x_2 = 2 \), \( x_1 + 2(-2) + 4 = 3 \Rightarrow x_1 = 3 \)

\((x_1, x_2, x_3) = (3, -2, 4) \) is the solution.