REM ARKS There are 6 problems. Each of the first two problems is worth 16 points. Each of the remaining four problems is worth 17 points. Show all relevant work on your exam. Be sure that your name is written on the test. NO CALCULATORS.

1.

a. Compute the Wronskian of the pair $e^{2t}, e^{-3t/2}$.

$$\begin{vmatrix}
    e^{2t} & e^{-3t/2} \\
    2e^{2t} & -3/2 e^{-3t/2}
\end{vmatrix} = \frac{-7}{2} e^{t/2}$$

b. Given the ODE $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$, compute the Wronskian of two fundamental solutions to this equation without solving the equation.

$$W = C e^{-\int \frac{-2x}{1-x^2} dx} = C e^{-\int \frac{du}{u}} = \frac{C}{1-x^2}$$

2.

a. Write the general solution to the equation $y'' + 4y' + 5y = 0$.

$$\gamma = -2 \pm i\sqrt{3}$$

$$y(t) = C_1 e^{-2t} \cos t + C_2 e^{-2t} \sin t$$

b. Solve the equation in a. along with the boundary conditions $y(0) = 1$, $y'(0) = 0$.

$$y(0) = 1 = C_1$$

$$y'(0) = 0 = -2C_2 + C_1$$

$$C_2 = 2$$
3.

a. Find the **general** solution to \( y'' + 2y' + y = 0 \).

\[(r+1)^2 = 0 \]

\[y(t) = c_1 e^{-t} + c_2 te^{-t} \]

b. Using the **method of undetermined coefficients** find a **particular** solution of

\[y'' + 2y' + y = 2e^{-t} \]

\[y(t) = c_2 te^{-t} \]

4. Given the second order equation

\[(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}, \quad 0 < t < 1 \]

a. Verify that \( y_1(t) = e^t \) and \( y_2(t) = t \) are solutions for the homogeneous version of this equation.

b. Use the method of **Variation of Parameters** to solve the nonhomogeneous equation.

\[ y'' + \left( \frac{t}{1-t} \right) y' + \left( \frac{1}{1-t} \right) y = 2(1-t)e^{-t} = g(t) \]

\[ y = u_1 y_1 + u_2 y_2 \]

\[ y = u_1 e^t + u_2 t \]

\[ W = \begin{vmatrix} e^t & t e^t \\ e^t & e^t \end{vmatrix} = e^t(1-t) \]

\[ u_1' = -a t e^{-at} \]

\[ u_1' = \frac{ae^{-t}}{t e^{-at} + \frac{e^{-at}}{2}} \]

\[ u_2' = ae^{-t} \]

\[ u_2 = -ae^{-t} \]

\[ y(t) = \frac{1}{ae^{-t}} e^{-t} (1-at) \]
\[ y(s) = \mathcal{L}\{y\} \]

5. Solve the IVP
\[ y'' - y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = 0 \]
by means of Laplace Transform Methods.

\[ \mathcal{L}\{y''\} - \mathcal{L}\{y'\} - 2\mathcal{L}\{y\} = 0 \]
\[ (s^2 - s - 2)\mathcal{L}\{y\} + (s - 2)\mathcal{L}\{y\} - \mathcal{L}\{y\}' = 0 \]
\[ \mathcal{L}\{y\} = \frac{s - 1}{(s - 2)(s - 1)} = \frac{1/3}{s - 2} + \frac{\sqrt{3}}{4 + 1} \]

\[ y(t) = \frac{1}{3} e^{2t} + \frac{\sqrt{3}}{3} e^{-t} \]

6.

a. Find the Laplace Transform of

[Formula #13, p. 317]
\[ f(t) = \begin{cases} 0, & \text{if } t < 2 \\ (t - 2)^2, & \text{if } t \geq 2 \end{cases} \]

\[ f(t) = u_2(t)g(t - 2) \]
\[ g(t) = t^2 \]
\[ f(s) = e^{-2s}G(s) \]

b. Find the Inverse Laplace Transform of \( F(s) = \frac{1}{(s - 2)^2} \).

[Formula #11, p. 317]
\[ t^n e^{-at} \rightarrow \frac{n!}{(s - a)^{n+1}} \]

\[ f(t) = t^3 e^{2t} \]