MA 304 EXAM 2 —— Fall 2013

REMARKS There are 8 problems. Problems 1-4 are each worth 13 points while problems 5-8 are each worth 12 points. Show all relevant work. NO CALCULATORS.

1.

a. Find a basis for the subspace $S$ of $R^4$ consisting of all vectors of the form

$$\begin{pmatrix} a+b, a-b+2c, b, c \end{pmatrix} = a \begin{pmatrix} 1, 1, 0, 0 \end{pmatrix} + b \begin{pmatrix} 1, -1, 1, 0 \end{pmatrix} + c \begin{pmatrix} 0, 2, 0, 1 \end{pmatrix}$$

where $a$, $b$, $c$ are all real numbers. What is the dimension of $S$?

b. Is it possible to find a pair of two-dimensional subspaces $U$ and $V$ of $R^3$ such that $U \cap V = \{0\}$? Prove your answer.

2.

a. Given

$$v_1 = \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 4 \\ 1 & 2 \end{pmatrix}$$

Find vectors $u_1$ and $u_2$ so that $S$ will be the transition matrix from $\{v_1, v_2\}$ to $\{u_1, u_2\}$.

b. Find the transition matrix representing the change of coordinates on $P_3$ from the ordered basis $[1, x, x^2]$ to the ordered basis $[1, 1 + x, 1 + x + x^2]$.

$$S = \begin{pmatrix} 6 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$S^{-1}$ is the answer.
3. Let \( x \) and \( y \) be nonzero vectors in \( R^m \) and \( R^n \), respectively, and let \( A = xy^T \) be an \( m \times n \) matrix.

   a. Show that \( \{x\} \) is a basis for the column space of \( A \) and that \( \{y^T\} \) is a basis for the row space of \( A \).

   \[
   A = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} (y_1, \ldots, y_n)_{x \times n}
   \]
   \[
   A_{jk} = x_j y_k \quad \text{and} \quad \text{row vector is } x_j (y_1, \ldots, y_n)
   \]

   b. What is the dimension of the null space of \( A \)?

   \[
   \text{nullity rank theorem:} \quad \text{nullity } + \text{rank } = \text{dim } R^n
   \]
   \[
   A: R^n \rightarrow R^m \quad \Rightarrow \quad \text{nullity } + \text{rank } = \text{dim } R^n
   \]
   \[
   \text{rank } = 1 \quad \Rightarrow \quad \text{dim nullspace is } \boxed{n - 1}
   \]

4. Let \( A \) be an \( 6 \times n \) matrix of rank \( r \) and let \( b \) be a vector in \( R^6 \). For each pair of values of \( r \) and \( n \) that follow, indicate the possibilities as to the number of solutions one could have for the linear system \( Ax = b \). Explain your answers.

   a. \( n = 7, r = 5 \)
   \[
   \begin{align*}
   &\text{either } 1) \text{ no solution since } r < 6 \\
   &\quad 2) \text{ or } \infty \text{ many solutions since } \text{rank } = 5 \\
   &\text{null } = 7
   \end{align*}
   \]

   b. \( n = 7, r = 6 \)
   \[
   \infty \text{ many solutions } \begin{align*}
   &\text{since } r = 6 \text{ and } \text{null } = 1 \\
   \end{align*}
   \]

   c. \( n = 5, r = 5 \)
   \[
   \begin{align*}
   &\text{either } 1) \text{ no solution since } r < 5 \\
   &\quad \text{or } 2) \text{ unique solution since } \text{null } = 0 \\
   &\text{5 equations}
   \end{align*}
   \]

   d. \( n = 5, r = 4 \)
   \[
   \begin{align*}
   &\text{either } 1) \text{ no solution } r = 4 < 6 \\
   &\quad 2) \text{ or } \infty \text{ many solutions } \text{null } = 1 - 4
   \end{align*}
   \]

2
5. Find the kernel and range of each of the following linear operators on $P_3$:

a. $L(p(x)) = xp'(x)$.

$$L(a+bx+cx^2) = x(a+bx+cx^2)$$

Kernel $L = \{ \alpha \neq 0 \}$

Range $L = \mathbb{R}^2$

b. $L(p(x)) = p(x) - p'(x)$

$$L(a+bx+cx^2) = -cx^2+bx+a-b$$

Kernel $L = \{ \alpha \neq 0 \}$, Range $L = \mathbb{R}^3$

c. $L(p(x)) = p(0)x + p(1)$.

$$L(a+bx+cx^2) = ax + (a+bx+c)$$

Kernel $L = \{ 3x-x^2 \}$

Range $L = \mathbb{R}$

6. Let $S$ be the subspace of $C[a, b]$ spanned by $e^x$, $xe^x$, and $x^2e^x$. Let $D$ be the differentiation operator of $S$. Find the matrix representing $D$ with respect to $[e^x, xe^x, x^2e^x]$.

$$D e^x = e^x$$

$$D xe^x = e^x + xe^x$$

$$D x^2e^x = 2xe^x + x^2e^x$$

$$e^x \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Matrix of $D$ with respect to the given basis
7. Let $L$ be the linear operator mapping $\mathbb{R}^3$ into $\mathbb{R}^3$ defined by $L(x) = Ax$, where

$$A = \begin{pmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{pmatrix}$$

and let

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

Find the transition matrix $V$ corresponding to a change of basis from $\{v_1, v_2, v_3\}$ to $\{e_1, e_2, e_3\}$ and use it to determine the matrix $B$ representing $L$ with respect to the basis $\{v_1, v_2, v_3\}$.

$$V = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \quad V^{-1} A V = B$$

8.

a. Find the point on the line $y = 2x + 1$ that is closest to the point $(5, 2)$.

$$\mathbf{v} = (5-x, 1-2x) \quad (1, 2) = 0$$

$$5 - x + 2 - 4x = 7 - 5x = 0$$

$$x = \frac{7}{5}$$

$$y = 2x + 1 = \frac{19}{5}$$

b. Find the distance from the point $(1, 1, 1)$ to the plane $2x + 2y + z = 0$.

$$d = \left| \frac{(1, 1, 1) \cdot (2, 2, 1)}{3} \right| = \frac{5}{3}$$