1.2 Elementary Functions

1.2.1 What Is a Function?

Scientific investigations often study relationships between quantities, such as how enzyme activity depends on temperature or how the length of a fish is related to its age. To describe such relationships mathematically, the concept of a function is useful.

The word function (or more precisely, its Latin equivalent functio, which means "execution") was introduced by Leibniz in 1694 in order to describe curves. Later, Euler (1707-1783) used it to describe any equation involving variables and constants. The modern definition is much broader and emphasizes the basic idea of expressing relationships between any two sets.

**Definition** A function \( f \) is a rule that assigns each element \( x \) in the set \( A \) exactly one element \( y \) in the set \( B \). The element \( y \) is called the **image** (or value) of \( x \) under \( f \) and is denoted by \( f(x) \) (read "\( f \) of \( x \)"). The set \( A \) is called the domain of \( f \), the set \( B \) is called the codomain of \( f \), and the set \( f(A) = \{ y : y = f(x) \text{ for some } x \in A \} \) is called the range of \( f \).

To define a function, we use the notation

\[
 f : A \to B \\
 x \mapsto f(x)
\]

where \( A \) and \( B \) are subsets of the set of real numbers. Frequently, we simply write \( y = f(x) \) and call \( x \) the **independent** variable and \( y \) the **dependent** variable. We can illustrate functions graphically in the \( x-y \) plane. In Figure 1.9, we see the graph of \( y = f(x) \), with domain \( A \), codomain \( B \), and range \( f(A) \).

The function \( f(x) \) must be specified; for example, \( f(x) \) could be given by a graph as in Figure 1.9, or it could be expressed algebraically, such as \( f(x) = x^2 \). Note that \( f(A) \subseteq B \), but not every element in the codomain \( B \) must be in \( f(A) \). For instance, let

\[
 f : \mathbb{R} \to \mathbb{R} \\
 x \mapsto x^2
\]

The domain of \( f \) is \( \mathbb{R} \), but the range of \( f \) is only \([0, \infty)\) because the square of a real number is nonnegative; that is, \( f(\mathbb{R}) = [0, \infty) \neq \mathbb{R} \). Also, the domain of a function need not be the largest possible set on which we can define the function, as \( \mathbb{R} \) is in the preceding example. For instance, we could have defined \( f \) on a smaller set, such as \([0, 1]\), calling the new function \( g \), given by

\[
 g : [0, 1] \to \mathbb{R} \\
 x \mapsto x^2
\]

Although the same rule is used for \( f \) and \( g \), the two functions are not the same, because their respective domains are different.

**Example 1**

Two functions \( f \) and \( g \) are equal if and only if

1. \( f \) and \( g \) are defined on the same domain, and
2. \( f(x) = g(x) \) for all \( x \) in the domain.

Let

\[
 f_1 : [0, 1] \to \mathbb{R} \\
 x \mapsto x^2
\]

\[
 f_2 : [0, 1] \to \mathbb{R} \\
 x \mapsto \sqrt{x^4}
\]
and

\[ f_3 : \mathbb{R} \rightarrow \mathbb{R} \]
\[ x \rightarrow x^2 \]

Determine which of these functions are equal.

Because \( f_1 \) and \( f_2 \) are defined on the same domain and \( f_1(x) = f_2(x) = x^2 \) for all \( x \in [0, 1] \), it follows that \( f_1 \) and \( f_2 \) are equal.

Neither \( f_1 \) nor \( f_2 \) is equal to \( f_3 \), because the domain of \( f_3 \) is different from the domains of \( f_1 \) and \( f_2 \).

The choices of domains for the functions that we have thus far considered may look somewhat arbitrary (and they are arbitrary in the examples we have seen so far). In applications, however, there is often a natural choice of domain. For instance, if we look at a certain plant response (such as total biomass or the ratio of above to below biomass) as a function of nitrogen concentration in the soil, then, given that nitrogen concentration cannot be negative, the domain for this function could be the set of nonnegative real numbers. As another example, suppose we define a function that depends on the fraction of a population infected with a certain virus; then a natural choice for the domain of this function would be the interval \([0, 1]\) because a fraction of a population must be a number between 0 and 1.

In our definition of a function, we stated that a function is a rule that assigns, to each element \( x \in A \), exactly one element \( y \in B \). When we graph \( y = f(x) \) in the \( x-y \) plane, there is a simple test to decide whether or not \( f(x) \) is a function: If each vertical line intersects the graph of \( y = f(x) \) at most once, then \( f(x) \) is a function. Figure 1.10 shows the graph of a function. Each vertical line intersects the graph of \( y = f(x) \) at most once. The graph of \( y = f(x) \) in Figure 1.11 is not a function, since there are \( x \)-values that are assigned to more than one \( y \)-value, as illustrated by the vertical line that intersects the graph more than once.

Sometimes functions show certain symmetries. For example, in Figure 1.12, \( f(x) = x \) is symmetric about the origin; that is, \( f(x) = -f(-x) \). In Figure 1.13, \( g(x) = x^2 \) is symmetric about the \( y \)-axis; that is, \( g(x) = g(-x) \). In the first case, we say that \( f \) is odd; in the second case, that \( g \) is even. To check whether a function is even or odd, we use the following definition:

A function \( f : A \rightarrow B \) is called

1. **even** if \( f(x) = f(-x) \) for all \( x \in A \), and
2. **odd** if \( f(x) = -f(-x) \) for all \( x \in A \).
Using this criterion, we can show that \( f(x) = x, x \in \mathbb{R} \), is an odd function:

\[-f(-x) = -(-x) = x = f(x) \quad \text{for all } x \in \mathbb{R}\]

Likewise, to show that \( g(x) = x^2, x \in \mathbb{R} \), is an even function, we compute

\[g(-x) = (-x)^2 = x^2 = g(x) \quad \text{for all } x \in \mathbb{R}\]

We will now look at the case where one quantity is given as a function of another quantity that, in turn, can be written as a function of yet another quantity. To illustrate this situation, suppose we are interested in the abundance of a predator, which depends on the abundance of a herbivore, which, in turn, depends on the abundance of plant biomass. If we denote the plant biomass by \( x \) and the herbivore biomass by \( u \), then \( x \) and \( u \) are related via a function \( g \), namely, \( u = g(x) \). Likewise, if we denote the predator biomass by \( y \), then \( u \) and \( y \) are related via a function \( f \), namely, \( y = f(u) \). We can express the predator biomass as a function of the plant biomass by substituting \( g(x) \) for \( u \). That is, we find \( y = f(g(x)) \). Functions that are defined in such a way are called composite functions.

**Definition** The composite function \( f \circ g \) (also called the composition of \( f \) and \( g \)) is defined as

\[(f \circ g)(x) = f[g(x)]\]

for each \( x \) in the domain of \( g \) for which \( g(x) \) is in the domain of \( f \).

The composition of functions is illustrated in Figure 1.14. We call \( g \) the inner function and \( f \) the outer function. The phrase “for each \( x \) in the domain of \( g \) for which \( g(x) \) is in the domain of \( f \)” is best explained with the use of Figure 1.14. In order to compute \( f(u) \), \( u \) needs to be in the domain of \( f \). But since \( u = g(x) \), we really require that \( g(x) \) be in the domain of \( f \) for the values of \( x \) we use to compute \( g(x) \).

If \( f(x) = \sqrt{x}, x \geq 0, \) and \( g(x) = x^2 + 1, x \in \mathbb{R}, \) find

(a) \( (f \circ g)(x) \) and \( (b) \ (g \circ f)(x) \).

(a) To find \( (f \circ g)(x) \), we set \( f(u) = \sqrt{u} \) and \( g(x) = x^2 + 1 \). Then

\[y = f(u) = f[g(x)] = f(x^2 + 1) = \sqrt{x^2 + 1}\]

To determine the domain of \( f \circ g \), we observe that the domain of the inner function \( g \) is \( \mathbb{R} \) and its range is \([1, \infty)\). Since the range of \( g \) is contained in the domain of the outer function \( f \) \((1, \infty) \subset [0, \infty))\), the domain of \( f \circ g \) is \( \mathbb{R} \).

(b) To find \( (g \circ f)(x) \), we set \( g(u) = u^2 + 1 \) and \( f(x) = \sqrt{x} \). Then

\[y = g(u) = g[f(x)] = g(\sqrt{x}) = (\sqrt{x})^2 + 1 = x + 1\]

To determine the domain of \( g \circ f \), we observe that the domain of the inner function \( f \) is \([0, \infty)\) and its range is \([0, \infty)\). The range of \( f \) is contained in the domain of the outer function \( g \) \((0, \infty) \subset \mathbb{R})\), so the domain of \( g \circ f \) is \([0, \infty)\).

In the last example, you should observe that \( f \circ g \) is different from \( g \circ f \), which implies that the order in which you compose functions is important. The notation \( f \circ g \) means that you apply \( g \) first and then \( f \). In addition, you should pay attention to the domains of composite functions. In the next example, the domain is harder to find.

**Example 3**

If \( f(x) = 2x^2, x \geq 2, \) and \( g(x) = \sqrt{x}, x \geq 0, \) find \( (f \circ g)(x) \) together with its domain.
We compute \((f \circ g)(x) = f[g(x)] = f(\sqrt{x}) = 2(\sqrt{x})^2 = 2x\).

This part was not difficult. However, finding the domain of \(f \circ g\) is more complicated. The domain of the inner function \(g\) is the interval \([0, \infty)\); hence, the range of \(g\) is the interval \([0, \infty)\). The domain of \(f\) is only \([2, \infty)\), which means that the range of \(g\) is not contained in the domain of \(f\). We therefore need to restrict the domain of \(g\) to ensure that its range is contained in the domain of \(f\). We can choose only values of \(x\) such that \(g(x) \in [2, \infty)\). Since \(g(x) = \sqrt{x}\), we need to restrict \(x\) to \([4, \infty)\). Thus, for every \(x \in [4, \infty)\), \(g(x) \in [2, \infty)\), which is the domain of \(f\). Therefore,

\[(f \circ g)(x) = 2x, \quad x \geq 4\]

See Figure 1.15.

In the subsections that follow, we introduce the basic functions that are used throughout the remainder of this book.

1.2.2 Polynomial Functions

Polynomial functions are the simplest elementary functions.

**Definition** A polynomial function is a function of the form

\[f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n\]

where \(n\) is a nonnegative integer and \(a_0, a_1, \ldots, a_n\) are (real-valued) constants with \(a_n \neq 0\). The coefficient \(a_n\) is called the leading coefficient, and \(n\) is called the degree of the polynomial function. The largest possible domain of \(f\) is \(\mathbb{R}\).

We have already encountered polynomials, namely, the constant function \(f(x) = c\), the linear function \(f(x) = mx + b\), and the quadratic function \(f(x) = ax^2\). The constant, nonzero function has degree 0, the linear function has degree 1, and the quadratic function has degree 2. Other examples are \(f(x) = 4x^3 - 3x + 1, \, x \in \mathbb{R}\), which is a polynomial of degree 3, and \(f(x) = 2 - x, \, x \in \mathbb{R}\), which is a polynomial of degree 1. In Figure 1.16, we display \(y = x^n\) for \(n = 2\) and 3. Looking at the figure, we see that \(y = x^n\) is an even function (i.e., symmetric about the y-axis) when \(n = 2\) and an odd function (i.e., symmetric about the origin) when \(n = 3\). This property holds in general: \(y = x^n\) is an even function when \(n\) is even and an odd function when \(n\) is odd. We can show this algebraically by using the criterion in Section 1.2.1. (See Problem 28 at the end of this section.)

![Figure 1.16 The graphs of \(y = x^n\) for \(n = 2\) and \(n = 3\).](image)

Polynomials arise naturally in many situations. We present two examples.
EXAMPLE 4  
Suppose that at time 0 an apple begins to drop from a tree that is 64 ft tall. Ignoring air resistance, we can show that at time \( t \) (measured in seconds) the apple is at height \( h(t) \) (measured in feet) given by

\[
h(t) = 64 - 16t^2
\]

We assume that the height of the ground level is equal to 0. Show that \( h(t) \) is a polynomial and determine its degree. How long will it take the apple to hit the ground? Find an appropriate domain for \( h(t) \).

\[\text{Figure 1.17 The graph of } h(t) = 64 - 16t^2 \text{ for } 0 \leq t \leq 2 \text{ of Example 4.}\]

Solution  
The function \( h(t) \) is a polynomial of degree 2, with \( a_0 = 64 \), \( a_1 = 0 \), and \( a_2 = -16 \). The graph of \( h(t) \) is shown in Figure 1.17. The apple will hit the ground when \( h(t) = 0 \). That is, we must solve the quadratic equation \( 0 = 64 - 16t^2 \) as follows:

\[
0 = 64 - 16t^2 \\
t^2 = \frac{64}{16} = 4 \\
t = 2 \quad \text{or} \quad t = -2
\]

Since the apple begins to drop at time \( t = 0 \), we can ignore the solution \( t = -2 < 0 \). We find that it takes the apple 2 seconds to hit the ground (ignoring air resistance). Note that because \( h(t) \geq 0 \) [where \( h(t) \) is the height above the ground and the height of the ground level is equal to 0], the range is \([0, 64]\). Because \( t \geq 0 \), the domain of \( h(t) \) is the interval \([0, 2]\).

EXAMPLE 5  
A Chemical Reaction  
Consider the reaction rate of the chemical reaction

\[
A + B \rightarrow AB
\]

in which the molecular reactants A and B form the molecular product AB. The rate at which this reaction proceeds depends on how often A and B molecules collide. The law of mass action states that the rate at which this reaction proceeds is proportional to the product of the respective concentrations of the reactants. Here, concentration means the number of molecules per fixed volume. If we denote the reaction rate by \( R \) and the concentration of A and B by \([A]\) and \([B]\), respectively, then the law of mass action says that

\[
R \propto [A] \cdot [B]
\]

Introducing the proportionality factor \( k \), we obtain

\[
R = k[A] \cdot [B]
\]
Note that $k > 0$, because $[A]$, $[B]$, and $R$ are positive. We assume now that the reaction occurs in a closed vessel; that is, we add specific amounts of $A$ and $B$ to the vessel at the beginning of the reaction and then let the reaction proceed without further additions.

We can express the concentrations of the reactants $A$ and $B$ during the reaction in terms of their initial concentrations $a$ and $b$ and the concentration of the molecular product $[AB]$. If $x = [AB]$, then

$$[A] = a - x \quad \text{for} \quad 0 \leq x \leq a \quad \text{and} \quad [B] = b - x \quad \text{for} \quad 0 \leq x \leq b$$

The concentration of $AB$ cannot exceed either of the concentrations of $A$ and $B$. (For example, suppose five $A$ molecules and seven $B$ molecules are allowed to react; then a maximum of five $AB$ molecules can result, at which point all of the $A$ molecules are used up and the reaction ceases. The two $B$ molecules left over have no $A$ molecules to react with.) Therefore, we get

$$R(x) = k(a - x)(b - x) \quad \text{for} \quad 0 \leq x \leq a \quad \text{and} \quad 0 \leq x \leq b$$

The condition $0 \leq x \leq a$ and $0 \leq x \leq b$ can be written as $0 \leq x \leq \min(a, b)$, where $\min(a, b)$ denotes the minimum of $a$ and $b$. To see that $R(x)$ is indeed a polynomial function, we expand the expression for $R(x)$ as

$$R(x) = k(ab - ax - bx + x^2)$$
$$= kx^2 - k(a + b)x + kab$$

for $0 \leq x \leq \min(a, b)$. We now see that $R(x)$ is a polynomial of degree 2.

A graph of $R(x)$, $0 \leq x \leq a$, is shown in Figure 1.18 for the case $a \leq b$. (We chose $k = 2$, $a = 2$, and $b = 5$ in the figure.) Notice that when $x = 0$ (i.e., when no $AB$ molecules have yet formed), the rate at which the reaction proceeds is at a maximum. As more and more $AB$ molecules form and, consequently, the concentrations of the reactants decline, the reaction rate decreases. This should also be intuitively clear: As fewer and fewer $A$ and $B$ molecules are in the vessel, it becomes less and less likely that they will collide to form the molecular product $AB$. When $x = a = \min(a, b)$, the reaction rate $R(a) = 0$. This is the point at which all $A$ molecules are exhausted and the reaction necessarily ceases.

![Figure 1.18](image_url)

**Figure 1.18** The graph of $R(x) = 2(2 - x)(5 - x)$ for $0 \leq x \leq 2$.

### 1.2.3 Rational Functions

Rational functions are built from polynomial functions.

**Definition** A rational function is the quotient of two polynomial functions $p(x)$ and $q(x)$:

$$f(x) = \frac{p(x)}{q(x)} \quad \text{for} \quad q(x) \neq 0$$
Since division by 0 is not allowed, we must exclude those values of \( x \) for which \( q(x) = 0 \). Here are a couple of examples of rational functions, together with their largest possible domains:

\[
y = \frac{1}{x}, \; x \neq 0
\]

\[
y = \frac{x^2 + 2x - 1}{x - 3}, \; x \neq 3
\]

An important example of a rational function is the hyperbola, together with its largest possible domain:

\[
y = \frac{1}{x}, \; x \neq 0
\]

The graph of \( y \) is shown in Figure 1.19.

![Figure 1.19 The graph of \( y = \frac{1}{x} \) for \( x \neq 0 \).](image)

Throughout this text, we will encounter populations whose sizes change with time. The change in population size is described by the growth rate. Roughly speaking, the growth rate tells you how much a population changes during a small time interval. (The growth rate is analogous to the velocity of a car: Velocity is also a rate; it tells you how much the position changes in a small time interval. We will give a precise definition of rates in Section 4.1.) The per capita growth rate is the growth rate divided by the population size. The per capita growth rate is also called the specific growth rate. The next example introduces a function that is frequently used to describe growth rates.

**Example 6**

**Monod Growth Function** There is a function that is frequently used to describe the per capita growth rate of organisms when the rate depends on the concentration of some nutrient and becomes saturated for large enough nutrient concentrations. If we denote the concentration of the nutrient by \( N \), then the per capita growth rate \( r(N) \) is given by the Monod growth function

\[
r(N) = \frac{aN}{k + N}, \quad N \geq 0
\]

where \( a \) and \( k \) are positive constants. The graph of \( r(N) \) is shown in Figure 1.20; it is a piece of a hyperbola. The graph shows a decelerating rise approaching the saturation level \( a \), which is the maximal specific growth rate. When \( N = k \), \( r(N) = a/2 \); for this reason, \( k \) is called the half-saturation constant. The growth rate increases with nutrient concentration \( N \); however, doubling the nutrient concentration has a much bigger effect on the growth rate for small values of \( N \) than when \( N \) is already large. When this type of function is used in biochemistry to describe enzymatic reactions, it is called the Michaelis–Menten function.

![Figure 1.20 The graph of the Monod function \( r(N) = \frac{a}{k + N} \) for \( N \geq 0 \).](image)
1.2.4 Power Functions

**Definition** A power function is of the form

\[ f(x) = x^r \]

where \( r \) is a real number.

Examples of power functions, with their largest possible domains, are

\[
\begin{align*}
  y &= x^{1/2}, \quad x \in \mathbb{R} \\
  y &= x^{5/2}, \quad x \geq 0 \\
  y &= x^{1/2}, \quad x \geq 0 \\
  y &= x^{-1/2}, \quad x > 0
\end{align*}
\]

Polynomials of the form \( y = x^n, \quad n = 1, 2, \ldots \), are a special case of power functions. Since power functions may involve even roots, as in \( y = x^{3/2} = (\sqrt{x})^3 \), we frequently need to restrict their domain.

Figure 1.21 compares the power functions \( y = x^{5/2}, \quad y = x^{1/2}, \) and \( y = x^{-1/2} \) for \( x > 0 \). Pay close attention to how the exponent determines the ranking according to size for \( x \) between 0 and 1 and for \( x > 1 \). We find that \( x^{5/2} < x^{1/2} < x^{-1/2} \) for \( 0 < x < 1 \), but \( x^{5/2} > x^{1/2} > x^{-1/2} \) for \( x > 1 \).

![Figure 1.21 Some power functions with rational exponents.](image)

**EXAMPLE 7**

Power functions are frequently found in "scaling relations" between biological variables (e.g., organ sizes). These are relations of the form

\[ y \propto x^r \]

where \( r \) is a nonzero real number. That is, \( y \) is proportional to some power of \( x \). Recall that we can write this relationship as an equation if we introduce the proportionality factor \( k \):

\[ y = kx^r \]

Finding such relationships is the objective of **allometry**. For example, in a study of 45 species of unicellular algae, a relationship between cell volume and cell biomass was sought. It was found [see, for instance, Niklas (1994)] that

\[ \text{cell biomass } \propto \text{ (cell volume)}^{0.704} \]

Most scaling relations are to be interpreted in a statistical sense; they are obtained by fitting a curve to data points. The data points are typically scattered about the fitted curve given by the scaling relation. (See Figure 1.22.)
The next example relates the volume and the surface area of a cube. This relationship is not to be understood in a statistical sense, because it is an exact relationship resulting from geometric considerations.

**Example 8**

Suppose that we wish to know the scaling relation between the surface area \( S \) and the volume \( V \) of a cube. The scaling relations of each of these quantities with the length \( L \) of the cube are as follows:

\[
S \propto L^2 \quad \text{or} \quad S = k_1 L^2
\]

\[
V \propto L^3 \quad \text{or} \quad V = k_2 L^3
\]

Here, \( k_1 \) and \( k_2 \) denote the constants of proportionality. (We label them with different subscripts to indicate that they might be different.) To express \( S \) in terms of \( V \), we must first solve \( L \) in terms of \( V \) and then substitute \( L \) in the equation for \( S \). Because \( L = (V/k_2)^{1/3} \), it follows that

\[
S = k_1 \left( \frac{V}{k_2} \right)^{1/3} = k_1 k_2^{1/3} \frac{V^{2/3}}{V^{2/3}}
\]

Introducing the constant of proportionality \( k = k_1 k_2^{-2/3} \), we find that

\[
S = k V^{2/3}, \quad \text{or simply} \quad S \propto V^{2/3}
\]

In words, the surface area of a cube scales with the volume in proportion to \( V^{2/3} \). We can now ask, for instance, by what factor the surface area increases when we double the volume. When we double the volume, we find that the resulting surface area, denoted by \( S' \), is

\[
S' = k(2V)^{2/3} = 2^{2/3} k V^{2/3}
\]

That is, the surface area increases by a factor of \( 2^{2/3} \approx 1.587 \) if we double the volume of the cube. This scaling has implications on heat retention in animals; a larger body has a relatively smaller surface area and will retain more heat.

### 1.2.5 Exponential Functions

In our study of exponential functions, let’s first look at an example that illustrates where they occur.
**Example 9**  

**Exponential Growth**  Bacteria reproduce asexually by cellular fission, in which the parent cell splits into two daughter cells after duplication of the genetic material. This division may happen as often as every 20 minutes; under ideal conditions, a bacterial colony can double in size in that time.

Let us measure time such that one unit of time corresponds to the doubling time of the colony. If we denote the size of the population at time $t$ by $N(t)$, then the function

$$N(t) = 2^t, \quad t \geq 0$$

has the property of doubling its value every unit of time

$$N(t + 1) = 2^{t+1} = 2 \cdot 2^t = 2N(t) \tag{1.2}$$

The function $N(t) = 2^t, t \geq 0$, is an exponential function because the variable $t$ is in the exponent. We call the number 2 the base of the exponential function $N(t) = 2^t$.

We find that when $t = 0$, $N(0) = 1$; that is, there is just one individual in the population at time $t = 0$. If, at time $t = 0$, 40 individuals were present in the population, we would write $N(0) = 40$ and

$$N(t) = 40 \cdot 2^t, \quad t \geq 0 \tag{1.3}$$

You can verify that $N(t)$ in (1.3) also satisfies $N(t + 1) = 2N(t)$.

It is often desirable not to specify the initial number of individuals in the equation describing $N(t)$. This approach has the advantage that the equation for $N(t)$ then describes a more general situation, in the sense that we can use the same equation for different initial population sizes. We often denote the population size at time 0 by $N_0$ (read “$N$ sub 0”) instead of $N(0)$. The equation for $N(t)$ is then

$$N(t) = N_0 2^t, \quad t \geq 0$$

We can verify that $N(0) = N_0 2^0 = N_0$ and that $N(t + 1) = N_0 2^{t+1} = 2(N_0 2^t) = 2N(t)$.

The function $f(t) = 2^t$ can be defined for all $t \in \mathbb{R}$; its graph is shown in Figure 1.23.

![Graph of the function $f(t) = 2^t$](image)

**Figure 1.23** The function $f(t) = 2^t, t \in \mathbb{R}$.

Here is the definition of an exponential function:

**Definition** The function $f$ is an exponential function with base $a$ if

$$f(x) = a^x$$

where $a$ is a positive constant other than 1. The largest possible domain of $f$ is $\mathbb{R}$. 

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When \( a = 1 \), \( f(x) = 1 \) for all values of \( x \). This is a case that will occur in biological examples, but is excluded from the definition since it is simply the constant function.

\[ f(x) = a^x \]

Figure 1.24  Exponential growth and exponential decay.

The basic shape of the exponential function \( f(x) = a^x \) depends on the base \( a \); two examples are shown in Figure 1.24. As \( x \) increases, the graph of \( f(x) = 2^x \) shows a rapid increase, whereas the graph of \( f(x) = (1/3)^x \) shows a rapid decrease toward 0. We find the rapid increase whenever \( a > 1 \) and the rapid decrease whenever \( 0 < a < 1 \). Therefore, we say that we have exponential growth when \( a > 1 \) and exponential decay when \( 0 < a < 1 \).

Recall that \( a^0 = 1 \) and \( a^{1/k} = \sqrt[k]{a} \), where \( k \) is a positive integer. In Subsection 1.1.5, we summarized the properties of exponentials. Since they are very important, we list them again here:

\[
\begin{align*}
  a^r a^s &= a^{r+s} \\
  \frac{a^r}{a^s} &= a^{r-s} \\
  a^{-r} &= \frac{1}{a^r} \\
  (a^r)^s &= a^{rs}
\end{align*}
\]

In many applications, the exponential function is expressed in terms of the base \( e = 2.718 \ldots \), which we encountered in Subsection 1.1.5. The number \( e \) is called the natural exponential base. The exponential function with base \( e \) is alternatively written as \( \exp(x) \). That is,

\[ \exp(x) = e^x \]

The advantage of this alternative form can be seen when we try to write something like \( e^{x^2/(x^2+1)} \): \( \exp(x^2/(x^2+1)) \) is easier to read. More generally, if \( g(x) \) is a function in \( x \), then we can write, equivalently,

\[ \exp[g(x)] \quad \text{or} \quad e^{g(x)} \]

Bases 2 and 10 are also frequently used; in calculus, however, \( e \) will turn out to be the most common base.

The next two examples provide an important application of exponential functions.

**Example 10**  Radioactive Decay  Radioactive isotopes such as carbon 14 are used to determine the absolute age of fossils or minerals, establishing an absolute chronology of the geological time scale. This technique was discovered in the early years of the 20th century and is based on the property of certain atoms to transform spontaneously by giving off protons, neutrons, or electrons. The phenomenon, called radioactive
decay, occurs at a constant rate that is independent of environmental conditions. The method was used, for instance, to trace the successive emergence of the Hawaiian islands, from the oldest, Kauai, to the youngest, Hawaii (which is about 100,000 years old).

Carbon 14 is formed high in the atmosphere. It is radioactive and decays into nitrogen (N\textsuperscript{14}). There is an equilibrium between atmospheric carbon 12 (C\textsuperscript{12}) and carbon 14 (C\textsuperscript{14})—a ratio that has been relatively constant over a fairly long period. When plants capture carbon dioxide (CO\textsubscript{2}) molecules from the atmosphere and build them into a product (such as cellulose), the initial ratio of C\textsuperscript{14} to C\textsuperscript{12} is the same as that in the atmosphere. Once the plants die, however, their uptake of CO\textsubscript{2} ceases, and the radioactive decay of C\textsuperscript{14} causes the ratio of C\textsuperscript{14} to C\textsuperscript{12} to decline. Because the law of radioactive decay is known, the change in ratio provides an accurate measure of the time since the plants’ death.

According to the radioactive decay law, if the amount of C\textsuperscript{14} at time \( t \) is denoted by \( W(t) \), with \( W(0) = W_0 \), then
\[
W(t) = W_0 e^{-\lambda t}, \quad t \geq 0
\]
where \( \lambda > 0 \) (\( \lambda \) is the lowercase Greek letter lambda) denotes the decay rate. The function \( W(t) = W_0 e^{-\lambda t} \) is another example of an exponential function. Its graph is shown in Figure 1.25.

Frequently, the decay rate is expressed in terms of the half-life of the material, which is the length of time that it takes for half of the material to decay. If we denote this time by \( T_h \), then (see Figure 1.25)
\[
W(T_h) = \frac{1}{2} W_0 = W_0 e^{-\lambda T_h}
\]
from which we obtain
\[
\frac{1}{2} = e^{-\lambda T_h}
\]
\[
2 = e^{\lambda T_h}
\]
Recall from algebra (or Subsection 1.1.5) that, to solve for the exponent \( \lambda T_h \), we must take logarithms on both sides. Since the exponent has base \( e \), we use natural logarithms and find that
\[
\ln 2 = \lambda T_h
\]
Solving for \( T_h \) or \( \lambda \) yields
\[
T_h = \frac{\ln 2}{\lambda} \quad \text{or} \quad \lambda = \frac{\ln 2}{T_h}
\]
It is known that the half-life of C\textsuperscript{14} is 5730 years. Hence,
\[
\lambda = \frac{\ln 2}{5730 \text{ years}}
\]
Note that the unit “years” appears in the denominator. It is important to carry the units along. When we compute \( \lambda t \) in the exponent of \( e^{-\lambda t} \), we need to measure \( t \) in units of years in order for the units to cancel properly. For example, suppose \( t = 2000 \) years; then
\[
\lambda t = \frac{\ln 2}{5730 \text{ years}} \times 2000 \text{ years} = \frac{(\ln 2)(2000)}{5730} \approx 0.2419
\]
and we see that “years” appears in both the numerator and the denominator and thus can be canceled.
An application of the $^{14}$C dating method is given in the next example.

**Example 11**

Suppose that, on the basis of their $^{12}$C content, samples of wood found in an archeological excavation site contain about 23% as much $^{14}$C as does living plant material. Determine when the wood was cut.

**Solution**

The ratio of the current amount of $^{14}$C to the amount of living plant material is expressed as

$$0.23 = \frac{W(t)}{W(0)} = e^{-\lambda t}$$

Taking logarithms (base $e$) on both sides, we obtain

$$\ln(0.23) = -\lambda t$$

or

$$\lambda t = -\ln(0.23) = \ln \frac{1}{0.23}$$

With $\lambda = \ln 2/(5730$ years) from Example 10,

$$t = \frac{5730\text{ years}}{\ln 2} \ln \frac{1}{0.23}$$

Using a calculator to compute this result, we find that the wood was cut about 12,150 years ago.

---

**1.2.6 Inverse Functions**

Before we can introduce logarithmic functions, we must understand the concept of inverse functions. Roughly speaking, the inverse of a function $f$ reverses the effect of $f$. That is, if $f$ maps $x$ into $y = f(x)$, then the inverse function, denoted by $f^{-1}$ (read “$f$ inverse”), takes $y$ and maps it back into $x$. (See Figure 1.26.) Not every function has an inverse. Because an inverse function is a function itself, we require that every value $y$ in the range of $f$ be mapped into exactly one value $x$. In other words, for a function to have an inverse, it must be true that whenever $x_1 \neq x_2$, it follows that $f(x_1) \neq f(x_2)$ or, equivalently, that $f(x_1) = f(x_2)$ implies $x_1 = x_2$. (Recall the definition of a function, in which we required that each element in the domain be assigned to exactly one element in the range.)

Functions that have the property “$x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$” [or, equivalently, “$f(x_1) = f(x_2)$ implies $x_1 = x_2$”] are called **one to one**. If you know what the graph of a particular function looks like over its domain, then it is easy to determine whether or not the function is one to one: If no horizontal line intersects the graph of the function $f$ more than once, then $f$ is one to one. This criterion is called the horizontal line test. We illustrate it in Figures 1.27 and 1.28.
Now consider $y = x^3$ and $y = x^2$, for $x \in \mathbb{R}$. The function $y = x^3$, $x \in \mathbb{R}$, has an inverse function, because $x_1^3 \neq x_2^3$ whenever $x_1 \neq x_2$. (See Figure 1.27.) The function $y = x^2$, $x \in \mathbb{R}$, does not have an inverse function, because $x_1 \neq x_2$ does not imply $x_1^2 \neq x_2^2$ (or, equivalently, $x_1^2 = x_2^2$ does not imply $x_1 = x_2$; see Figure 1.28).

The equation $x_1^3 = x_2^3$ implies only that $|x_1| = |x_2|$. Since $x_1$ and $x_2$ can be positive or negative, we cannot simply drop the absolute-value signs. For instance, both $-2$ and $2$ are mapped into $4$, and we find that $f(-2) = f(2)$ but $-2 \neq 2$. (Note that $|-2| = |2|$.) To invert this function, we would have to map $4$ into $-2$ and $2$, but then it would no longer be a function by our definition. By restricting the domain of $y = x^2$ to, say, $x \geq 0$, we can define an inverse function of $y = x^2$, $x \geq 0$.

Here is the formal definition of an inverse function:

**Definition** Let $f : A \rightarrow B$ be a one-to-one function with range $f(A)$. The inverse function $f^{-1}$ has domain $f(A)$ and range $A$ and is defined by

$$f^{-1}(y) = x \quad \text{if and only if} \quad y = f(x)$$

for all $y \in f(A)$.

Find the inverse function of $f(x) = x^3 + 1$, $x \geq 0$.

First, note that $f(x)$ is one to one. To see this quickly, graph the function and apply the horizontal line test. (See Figure 1.29.) Be aware, though, that unless you know what the graph looks like over its entire domain, the graphical approach can be misleading. To demonstrate it algebraically, start with $f(x_1) = f(x_2)$ and show that this implies $x_1 = x_2$:

$$f(x_1) = f(x_2)$$

$$x_1^3 + 1 = x_2^3 + 1$$

$$x_1^3 = x_2^3$$

Taking the third root on both sides gives $x_1 = x_2$, which tells us that $f(x)$ has an inverse. Now we will find $f^{-1}$.

To find an inverse function, we follow three steps:

1. **Write** $y = f(x)$:

   $$y = x^3 + 1$$

2. **Solve for** $x$:

   $$x^3 = y - 1$$

   $$x = \sqrt[3]{y - 1}$$

   The range of $f$ is $[1, \infty)$, and this range becomes the domain of $f^{-1}$, so we obtain

   $$f^{-1}(y) = \sqrt[3]{y - 1}, \quad y \geq 1$$

   Typically, we write functions in terms of $x$. To do this, we need to interchange $x$ and $y$ in $x = f^{-1}(y)$. This is the third step:

3. **Interchange** $x$ and $y$:

   $$y = f^{-1}(x) = \sqrt[3]{x - 1}, \quad x \geq 1$$

   Note that switching $x$ and $y$ in step 3 corresponds to reflecting the graph of $y = f(x)$ about the line $y = x$. The graphs of $f$ and $f^{-1}$ are shown in Figure 1.30. Look at the graphs carefully, and observe how they are related to each other. Each can be obtained from the other by reflection about the line $y = x$. 

\[\square\]
As mentioned in the beginning of this subsection, the inverse of a function $f$ reverses the effect of $f$. If we first apply $f$ to $x$ and then $f^{-1}$ to $f(x)$, we obtain the original value $x$. Likewise, if we first apply $f^{-1}$ to $x$ and then $f$ to $f^{-1}(x)$, we obtain the original value $x$. That is, if $f : A \to B$ has an inverse $f^{-1}$, then

$$f^{-1}[f(x)] = x \quad \text{for all } x \in A$$
$$f[f^{-1}(x)] = x \quad \text{for all } x \in f(A)$$

[A note of warning: The superscript in $f^{-1}$ does not indicate the reciprocal of $f$ (i.e., $1/f$). This difference is further explained in Problem 74 at the end of this section.]

### 1.2.7 Logarithmic Functions

Recall from algebra (or Subsection 1.1.5) that, to solve the equation

$$e^x = 3$$

for $x$, you must take logarithms on both sides:

$$x = \ln 3$$

In other words, applying the natural logarithm undoes the operation of raising $e$ to the $x$ power. Thus, the natural logarithm is the inverse of the exponential function, and conversely, the exponential function is the inverse of the logarithmic function.

We will now define the inverse of the exponential function $f(x) = a^x$, $x \in \mathbb{R}$. The base $a$ can be any positive number, except 1.

**Definition** The inverse of $f(x) = a^x$ is called the logarithm to base $a$ and is written $f^{-1}(x) = \log_a x$.

The maximum domain of $f(x) = a^x$ is the set of all real numbers, and its range is the set of all positive numbers. Since the range of $f$ is the domain of $f^{-1}$, we find that the maximum domain of $f^{-1}(x) = \log_a x$ is the set of positive numbers.

Because $y = \log_a x$ is the inverse function of $y = a^x$, we can find the graph of $y = \log_a x$ by reflecting the graph of $y = a^x$ about the line $y = x$. Recall that the graph of $y = a^x$ had two basic shapes, depending on whether $0 < a < 1$ or $a > 1$. (See Figure 1.24.) Figure 1.31 illustrates the graphs of $y = a^x$ and $y = \log_a x$ when $a > 1$.

![Figure 1.31 The graph of $y = a^x$ and the graph of $y = \log_a x$ for $a = 2$.](image)

![Figure 1.32 The graph of $y = a^x$ and the graph of $y = \log_a x$ for $a = \frac{1}{2}$.](image)

Figure 1.32 shows the graphs of $y = a^x$ and $y = \log_a x$ when $0 < a < 1$. 
We can now summarize the relationship between the exponential and the logarithmic functions:

1. \(a^{\log_a x} = x\) for \(x > 0\)
2. \(\log_a a^x = x\) for \(x \in \mathbb{R}\)

It is important to remember that the logarithm is defined only for positive numbers; that is, \(y = \log_a x\) is defined only for \(x > 0\). The logarithm satisfies the following properties:

\[
\log_a (st) = \log_a s + \log_a t
\]
\[
\log_a \left(\frac{s}{t}\right) = \log_a s - \log_a t
\]
\[
\log_a s^r = r \log_a s
\]

The inverse of the exponential function with the natural base \(e\) is denoted by \(\ln x\) and is called the natural logarithm of \(x\). The graphs of \(y = e^x\) and \(y = \ln x\) are shown in Figure 1.33. Note that both \(e^x\) and \(\ln x\) are increasing functions. However, whereas \(e^x\) climbs very quickly for large values of \(x\), \(\ln x\) increases very slowly for large values of \(x\). Looking at both graphs, we can see that each can be obtained as the reflection of the other about the line \(y = x\).

The logarithm to base 10 is frequently written as \(\log x\) (i.e., the base of 10 in \(\log_{10} x\) is omitted).

**Example 13**

Simplify the following expressions:

(a) \(\log_2 [8(x - 2)]\)  (b) \(\log_3 9^x\)  (c) \(\ln e^{3x^2 + 1}\)

(a) We simplify as follows:

\[\log_2 [8(x - 2)] = \log_2 8 + \log_2 (x - 2) = 3 + \log_2 (x - 2)\]

No further simplification is possible.

(b) Simplifying yields

\[\log_3 9^x = x \log_3 9 = x \log_3 3^2 = 2x\]

The fact that \(\log_3 9 = 2\) can be seen in two ways: We can write \(9 = 3^2\) and say that applying \(\log_3\) undoes raising 3 to the second power (as we did previously), or we can say that \(\log_3 9\) denotes the exponent to which we must raise 3 in order to get 9.

(c) We use the fact that \(\ln x\) and \(e^x\) are inverse functions and find that

\[\ln e^{3x^2 + 1} = 3x^2 + 1\]

Any exponential function with base \(a\) can be written as an exponential function with base \(e\). Likewise, any logarithm to base \(a\) can be written in terms of the natural logarithm. The following two identities show how:

\[a^x = \exp(x \ln a)\]
\[\log_a x = \frac{\ln x}{\ln a}\]

The first identity follows from the fact that \(\exp\) and \(\ln\) are inversely related (which implies that \(a^x = \exp(\ln a^x)\)) and the fact that \(\ln a^x = x \ln a\). To understand the second identity, note that

\[y = \log_a x\ means\ \ a^y = x\]
Taking logarithms to base $e$ on both sides of $a^y = x$, we get

$$\ln a^y = \ln x$$

or

$$y \ln a = \ln x$$

Hence,

$$y = \frac{\ln x}{\ln a}$$

**EXAMPLE 14**

Write the following expressions in terms of base $e$:

(a) $2^x$

(b) $10^{x+1}$

(c) $\log_3 x$

(d) $\log_2 (3x - 1)$

**Solution**

(a) $2^x = \exp(\ln 2^x) = \exp(x \ln 2) = e^{x \ln 2}$

(b) $10^{x+1} = \exp(\ln 10^{x+1}) = \exp [(x + 1) \ln 10] = e^{(x+1) \ln 10}$

(c) $\log_3 x = \frac{\ln x}{\ln 3}$

(d) $\log_2 (3x - 1) = \frac{\ln(3x - 1)}{\ln 2}$

**EXAMPLE 15**

DNA sequences evolve over time by various processes. One such process is the substitution of one nucleotide for another. The simplest substitution scheme is that of Jukes and Cantor (1969), which assumes that substitutions are equally likely among the four types of nucleotide. In comparing two DNA sequences that have a common origin, it is possible to estimate the number of substitutions per site. Since more than one substitution can occur per site, the number of observed substitutions may be smaller than the number of actual substitutions, particularly when the time of divergence is large. Mathematical models are used to correct for this difference. The proportion $p$ of observed nucleotide differences between two sequences that share a common ancestor can be used to find an estimate of the actual number $K$ of substitutions per site since the time of divergence. According to the substitution scheme of Jukes and Cantor, $K$ and $p$ are related by

$$K = -\frac{3}{4} \ln \left(1 - \frac{4}{3} p\right)$$

provided that $p$ is not too large. Assume that two sequences of length 150 nucleotides differ from each other by 23 nucleotides. Find $K$.

**Solution**

The variable $p$ denotes the proportion of observed nucleotide differences, which is $23/150 \approx 0.1533$ in this example. We thus obtain

$$K = -\frac{3}{4} \ln \left(1 - \frac{4}{3} \frac{23}{150}\right) \approx 0.1715$$

**1.2.8 Trigonometric Functions**

The trigonometric functions are examples of periodic functions.

**Definition** A function $f(x)$ is periodic if there is a positive constant $a$ such that

$$f(x + a) = f(x)$$

for all $x$ in the domain of $f$. If $a$ is the smallest number with this property, we call it the period of $f(x)$. 
We begin with the sine and cosine functions. In Subsection 1.1.4, we recalled the definition of sine and cosine on a unit circle. There, \( \sin \theta \) and \( \cos \theta \) represented trigonometric functions of angles, and \( \theta \) was measured in degrees or radians. Now we define the trigonometric functions as functions of real numbers. For instance, we define \( f(x) = \sin x \) for \( x \in \mathbb{R} \). The value of \( \sin x \) is then, by definition, the sine of an angle of \( x \) radians (and similarly for all the other trigonometric functions).

The graphs of the sine and cosine functions are shown in Figures 1.34 and 1.35, respectively.

The sine function, \( y = \sin x \), is defined for all \( x \in \mathbb{R} \). Its range is \(-1 \leq y \leq 1\). Likewise, the cosine function, \( y = \cos x \), is defined for all \( x \in \mathbb{R} \) with range \(-1 \leq y \leq 1\). Both functions are periodic with period \( 2\pi \). That is, \( \sin(x + 2\pi) = \sin x \) and \( \cos(x + 2\pi) = \cos x \). We also have \( \sin(x + 4\pi) = \sin x \), \( \sin(x + 6\pi) = \sin x \), \ldots, and \( \cos(x + 4\pi) = \cos x \), \( \cos(x + 6\pi) = \cos x \), \ldots, but, by convention, we use the smallest possible value to specify the period. We see from Figures 1.34 and 1.35 that the graph of the cosine function can be obtained by shifting the graph of the sine function a distance of \( \pi/2 \) units to the left. (We will discuss horizontal shifts of graphs in more detail in the next section.)

To define the tangent function, \( y = \tan x \), recall that

\[
\tan x = \frac{\sin x}{\cos x}
\]

Because \( \cos x = 0 \) for values of \( x \) that are odd integer multiples of \( \pi/2 \), the domain of \( \tan x \) consists of all real numbers with the exception of odd integer multiples of \( \pi/2 \). The range of \( y = \tan x \) is \(-\infty < y < \infty\). The graph of \( y = \tan x \) is shown in Figure 1.36, from which we see that \( \tan x \) is periodic with period \( \pi \).

The graphs of the remaining three trigonometric functions are shown in Figures 1.37-1.39. Recall that \( \sec x = \frac{1}{\cos x} \), \( \csc x = \frac{1}{\sin x} \), and \( \cot x = \frac{1}{\tan x} \). It follows that the domain of the secant function \( y = \sec x \) consists of all real numbers with the exception of odd integer multiples of \( \pi/2 \); the range is \(|y| \geq 1\). The domain of the cosecant function \( y = \csc x \) consists of all real numbers with the exception of integer multiples of \( \pi \); the range is \(-\infty < y < -1 \) or \(1 < y < \infty \).

Since the sine and cosine functions are of particular importance, we now describe them in more detail. Consider the function

\[
f(x) = a \sin(kx)
\]

where \( a \) is a real number and \( k \neq 0 \). Now, \( f(x) \) takes on values between \(-a\) and \( a \). We call \(|a|\) the amplitude. The function \( f(x) \) is periodic. To find the period \( p \) of \( f(x) \), we set

\[
|k|p = 2\pi \quad \text{or} \quad p = \frac{2\pi}{|k|}
\]

Because the cosine function can be obtained from the sine function by a horizontal shift, we can define the amplitude and period analogously for the cosine function. That is, \( f(x) = a \cos(kx) \) has amplitude \(|a|\) and period \( p = 2\pi/|k| \).
EXAMPLE 16

Compare

\[ f(x) = 3 \sin \left( \frac{\pi x}{4} \right) \quad \text{and} \quad g(x) = \sin x \]

Solution

The amplitude of \( f(x) \) is 3, whereas the amplitude of \( g(x) \) is 1. The period \( p \) of \( f(x) \) satisfies \( \frac{\pi}{4} p = 2\pi \) or \( p = 8 \), whereas the period of \( g(x) \) is \( 2\pi \). Graphs of \( f(x) \) and \( g(x) \) are shown in Figure 1.40.

![Graph](image)

Figure 1.40 The graphs of \( y = 3 \sin \left( \frac{\pi}{4} x \right) \) and \( g(x) = \sin x \) in Example 16.

Remark. A number is called algebraic if it is the solution of a polynomial equation with rational coefficients. For instance, \( \sqrt{2} \) is algebraic, as it satisfies the equation \( x^2 - 2 = 0 \). Numbers that are not algebraic are called transcendental. For instance, \( \pi \) and \( e \) are transcendental.

A similar distinction is made for functions. We call a function \( y = f(x) \) algebraic if it is the solution of an equation of the form

\[ P_n(x)y^n + \cdots + P_1(x)y + P_0(x) = 0 \]

in which the coefficients are polynomial functions in \( x \) with rational coefficients. For instance, the function \( y = 1/(1 + x) \) is algebraic, as it satisfies the equation \( (x + 1)y - 1 = 0 \). Here, \( P_1(x) = x + 1 \) and \( P_0(x) = -1 \). Other examples of algebraic functions are polynomial functions with rational coefficients and rational functions with rational coefficients.

Functions that are not algebraic are called transcendental. All the trigonometric, exponential, and logarithmic functions that we introduced in this section are transcendental functions.

Section 1.2 Problems

1.2.1

In Problems 1–4, state the range for the given functions. Graph each function.

1. \( f(x) = x^2, x \in \mathbb{R} \)
2. \( f(x) = x^3, x \in [0, 1] \)
3. \( f(x) = x^2, -1 < x \leq 0 \)
4. \( f(x) = x^2, -\frac{1}{2} < x < \frac{1}{2} \)

5. (a) Show that, for \( x \neq 1 \),

\[ \frac{x^2 - 1}{x - 1} = x + 1 \]

(b) Are the functions

\[ f(x) = \frac{x^2 - 1}{x - 1}, \quad x \neq 1 \]

and

\[ g(x) = x + 1, \quad x \in \mathbb{R} \]

equal?

6. (a) Show that

\[ 2|x - 1| = \begin{cases} 2(x - 1) & \text{for } x \geq 1 \\ 2(l - x) & \text{for } x \leq 1 \end{cases} \]
(b) Are the functions
\[ f(x) = \begin{cases} 
2 - 2x & \text{for } 0 \leq x \leq 1 \\
2x - 2 & \text{for } 1 \leq x \leq 2
\end{cases} \]
and
\[ g(x) = 2|x - 1|, \quad x \in [0, 2] \]
equal?

In Problems 7–12, sketch the graph of each function and decide in each case whether the function is (i) even, (ii) odd, or (iii) does not show any obvious symmetry. Then use the criteria in Subsection 1.2.1 to check your answers.

7. \( f(x) = 2x \)
8. \( f(x) = 3x^2 \)
9. \( f(x) = |3x| \)
10. \( f(x) = 2x + 1 \)
11. \( f(x) = -|x| \)
12. \( f(x) = 3x^3 \)

13. Suppose that
\[ f(x) = x^2, \quad x \in \mathbb{R} \]
and
\[ g(x) = 3 + x, \quad x \in \mathbb{R} \]
(a) Show that
\[ (f \circ g)(x) = (3 + x)^2, \quad x \in \mathbb{R} \]
(b) Show that
\[ (g \circ f)(x) = 3 + x^2, \quad x \in \mathbb{R} \]

14. Suppose that
\[ f(x) = x^3, \quad x \in \mathbb{R} \]
and
\[ g(x) = 1 - x, \quad x \in \mathbb{R} \]
(a) Show that
\[ (f \circ g)(x) = (1 - x)^3, \quad x \in \mathbb{R} \]
(b) Show that
\[ (g \circ f)(x) = 1 - x^3, \quad x \in \mathbb{R} \]

15. Suppose that
\[ f(x) = 1 - x^2, \quad x \in \mathbb{R} \]
and
\[ g(x) = 2x, \quad x \geq 0 \]
(a) Find
\[ (f \circ g)(x) \]
together with its domain.
(b) Find
\[ (g \circ f)(x) \]
together with its domain.

16. Suppose that
\[ f(x) = \frac{1}{x+1}, \quad x \neq -1 \]
and
\[ g(x) = 2x^2, \quad x \in \mathbb{R} \]
(a) Find \((f \circ g)(x)\).
(b) Find \((g \circ f)(x)\).
In both (a) and (b), find the domain.

17. Suppose that
\[ f(x) = 3x^2, \quad x \geq 3 \]
and
\[ g(x) = \sqrt{x}, \quad x \geq 0 \]
Find \((f \circ g)(x)\) together with its domain.

18. Suppose that
\[ f(x) = x^4, \quad x \geq 3 \]
and
\[ g(x) = \sqrt{x + 1}, \quad x \geq 3 \]
Find \((f \circ g)(x)\) together with its domain.

19. Suppose that \( f(x) = x^2, \quad x \geq 0 \), and \( g(x) = \sqrt{x}, \quad x \geq 0 \). Typically, \( f \circ g \neq g \circ f \), but this is an example in which the order of composition does not matter. Show that \( f \circ g = g \circ f \).

20. Suppose that \( f(x) = x^4, \quad x \geq 0 \). Find \( g(x) \) so that \( f \circ g = g \circ f \).

1.2.2

21. Use a graphing calculator to graph \( f(x) = x^2, \quad x \geq 0 \), and \( g(x) = x^3, \quad x \geq 0 \), together. For which values of \( x \) is \( f(x) > g(x) \), and for which is \( f(x) < g(x) \)?

22. Use a graphing calculator to graph \( f(x) = x^2, \quad x \geq 0 \), and \( g(x) = x^3, \quad x \geq 0 \), together. When is \( f(x) > g(x) \), and when is \( f(x) < g(x) \)?

23. Graph \( y = x^n, \quad x \geq 0 \), for \( n = 1, 2, 3 \), and 4 in one coordinate system. Where do the curves intersect?

24. (a) Graph \( f(x) = x, \quad x \geq 0 \), and \( g(x) = x^3, \quad x \geq 0 \), together, in one coordinate system.
(b) For which values of \( x \) is \( f(x) \geq g(x) \), and for which values of \( x \) is \( f(x) \leq g(x) \)?

25. (a) Graph \( f(x) = x^2 \) and \( g(x) = x^3 \) for \( x \geq 0 \), together, in one coordinate system.
(b) Show algebraically that
\[ x^2 \geq x^3 \]
for \( 0 \leq x \leq 1 \).
(c) Show algebraically that
\[ x^2 \leq x^3 \]
for \( x \geq 1 \).

26. Show algebraically that if \( n \geq m \),
\[ x^n \leq x^m \]
for \( 0 \leq x \leq 1 \)
and
\[ x^n \geq x^m \]
for \( x \geq 1 \).

27. (a) Show that \( y = x^2, \quad x \in \mathbb{R} \), is an even function.
(b) Show that \( y = x^3, \quad x \in \mathbb{R} \), is an odd function.

28. Show that
(a) \( y = x^n, \quad x \in \mathbb{R} \), is an even function when \( n \) is an even integer.
(b) \( y = x^n, \quad x \in \mathbb{R} \), is an odd function when \( n \) is an odd integer.

29. In Example 5 of this section, we considered the chemical reaction

\[ A + B \rightarrow AB \]
Assume that initially only A and B are in the reaction vessel and that the initial concentrations are \( a = [A] = 3 \) and \( b = [B] = 4 \).
(a) We found that the reaction rate \( R(x) \), where \( x \) is the concentration of \( AB \), is given by

\[
R(x) = k(a - x)(b - x)
\]

where \( a \) is the initial concentration of \( A \), \( b \) is the initial concentration of \( B \), and \( k \) is the constant of proportionality. Suppose that the reaction rate \( R(x) \) is equal to \( 9 \) when the concentration of \( AB \) is \( x = 1 \). Use this relationship to find the reaction rate \( R(x) \).

(b) Determine the appropriate domain of \( R(x) \), and use a graphing calculator to sketch the graph of \( R(x) \).

30. An autocatalytic reaction uses its resulting product for the formation of a new product, as in the reaction

\[
A + X \rightarrow X
\]

If we assume that this reaction occurs in a closed vessel, then the reaction rate is given by

\[
R(x) = kx(a - x)
\]

for \( 0 \leq x \leq a \), where \( a \) is the initial concentration of \( A \) and \( x \) is the concentration of \( X \).

(a) Show that \( R(x) \) is a polynomial and determine its degree.

(b) Graph \( R(x) \) for \( k = 2 \) and \( a = 6 \). Find the value of \( x \) at which the reaction rate is maximal.

31. Suppose that a beetle walks up a tree along a straight line at a constant speed of 1 meter per hour. What distance will the beetle have covered after 1 hour, 2 hours, and 3 hours? Write an equation that expresses the distance (in meters) as a function of the time (in hours), and show that this function is a polynomial of degree 1.

32. Suppose that a fungal disease originates in the middle of an orchard, initially affecting only one tree. The disease spreads out radially at a constant speed of 10 feet per day. What area will be affected after 2 days, 4 days, and 8 days? Write an equation that expresses the affected area as a function of time, measured in days, and show that this function is a polynomial of degree 2.

1.2.3

In Problems 33–36, for each function, find the largest possible domain and determine the range.

33. \( f(x) = \frac{1}{1 - x} \)

34. \( f(x) = \frac{2x}{(x - 2)(x + 3)} \)

35. \( f(x) = \frac{x - 2}{x^2 - 9} \)

36. \( f(x) = \frac{1}{x^2 + 1} \)

37. Compare \( y = \frac{1}{x} \) and \( y = \frac{1}{x^2} \) for \( x > 0 \) by graphing the two functions. Where do the curves intersect? Which function is greater for small values of \( x \)? For large values of \( x \)?

38. Let \( n \) and \( m \) be two positive integers with \( m \leq n \). Answer the following questions about \( y = x^n \) and \( y = x^m \) for \( x > 0 \). Where do the curves intersect? Which function is greater for small values of \( x \)? For large values of \( x \)?

39. Let

\[
f(x) = \frac{1}{x + 1}, \quad x > -1
\]

(a) Use a graphing calculator to graph \( f(x) \).

(b) On the basis of the graph in (a), determine the range of \( f(x) \).

(c) For which values of \( x \) is \( f(x) = 2 \)?

(d) On the basis of the graph in (a), determine how many solutions \( f(x) = a \) has, where \( a \) is in the range of \( f(x) \).

40. Let

\[
f(x) = \frac{2x}{3 + x}, \quad x \geq 0
\]

(a) Use a graphing calculator to graph \( f(x) \).

(b) Find the range of \( f(x) \).

(c) For which values of \( x \) is \( f(x) = 1 \)?

(d) Based on the graph in (a), explain in words why, for any value \( a \) in the range of \( f(x) \), you can find exactly one value \( x \geq 0 \) such that \( f(x) = a \). Determine \( x \) by solving \( f(x) = a \).

41. Let

\[
f(x) = \frac{3x}{1 + x^2}, \quad x \geq 0
\]

(a) Use a graphing calculator to graph \( f(x) \).

(b) Find the range of \( f(x) \).

(c) For which values of \( x \) is \( f(x) = 2 \)?

(d) On the basis of the graph in (a), explain in words why, for any value \( a \) in the range of \( f(x) \), you can find exactly one value \( x \geq 0 \) such that \( f(x) = a \). Determine \( x \) by solving \( f(x) = a \).

In Problems 42–44, we discuss the Monod growth function, which was introduced in Example 6 of this section.

42. Use a graphing calculator to investigate the Monod growth function

\[
r(N) = \frac{aN}{k + N}, \quad N \geq 0
\]

where \( a \) and \( k \) are positive constants.

(a) Graph \( r(N) \) for (i) \( a = 5 \) and \( k = 1 \), (ii) \( a = 5 \) and \( k = 3 \), and (iii) \( a = 8 \) and \( k = 1 \). Place all three graphs in one coordinate system.

(b) On the basis of the graphs in (a), describe in words what happens when you change \( a \).

(c) On the basis of the graphs in (a), describe in words what happens when you change \( k \).

43. The Monod growth function \( r(N) \) describes growth as a function of nutrient concentration \( N \). Assume that

\[
r(N) = \frac{aN}{k + N}, \quad N \geq 0
\]

Find the percentage increase when the nutrient concentration is doubled from \( N = 0.1 \) to \( N = 0.2 \). Compare this result with what you find when you double the nutrient concentration from \( N = 10 \) to \( N = 20 \). This is an example of diminishing return.

44. The Monod growth function \( r(N) \) describes growth as a function of nutrient concentration \( N \). Assume that

\[
r(N) = \frac{aN}{k + N}, \quad N \geq 0
\]

where \( a \) and \( k \) are positive constants.

(a) What happens to \( r(N) \) as \( N \) increases? Use this relationship to explain why \( a \) is called the saturation level.

(b) Show that \( k \) is the half-saturation constant; that is, show that if \( N = k \), then \( r(N) = a/2 \).

45. Let

\[
f(x) = \frac{x^2}{4 + x^2}, \quad x \geq 0
\]

(a) Use a graphing calculator to graph \( f(x) \).

(b) On the basis of your graph in (a), find the range of \( f(x) \).

(c) What happens to \( f(x) \) as \( x \) gets larger?
46. The function 
\[ f(x) = \frac{x^n}{b^n + x^n}, \quad x \geq 0 \]
where \( n \) is a positive integer and \( b \) is a positive real number, is used in biochemistry to model reaction rates as a function of the concentration of some reactants. 

(a) Use a graphing calculator to graph \( f(x) \) for \( n = 1, 2, \) and \( 3 \) in one coordinate system when \( b = 2 \).

(b) Where do the three graphs in (a) intersect?

(c) What happens to \( f(x) \) as \( x \) gets larger?

(d) For an arbitrary positive value of \( b \), show that \( f(b) = 1/2 \). On the basis of this demonstration and your answer in (c), explain why \( b \) is called the half-saturation constant.

\[ \text{1.2.4} \]

In Problems 47–50, use a graphing calculator to sketch the graphs of the functions.

47. \( y = x^{3/2}, \quad x \geq 0 \)
48. \( y = x^{1/3}, \quad x \geq 0 \)
49. \( y = x^{-1/4}, \quad x > 0 \)
50. \( y = 2x^{-7/8}, \quad x > 0 \)

51. (a) Graph \( y = x^{-1/2}, \quad x > 0 \), and \( y = x^{1/2}, \quad x \geq 0 \), together, in one coordinate system.

(b) Show algebraically that
\[ x^{-1/2} \geq x^{1/2} \]
for \( 0 < x \leq 1 \).

(c) Show algebraically that
\[ x^{-1/2} \leq x^{1/2} \]
for \( x \geq 1 \).

52. (a) Graph \( y = x^{5/2}, \quad x \geq 0 \), and \( y = x^{3/2}, \quad x \geq 0 \), together, in one coordinate system.

(b) Show algebraically that
\[ x^{5/2} \leq x^{3/2} \]
for \( 0 \leq x \leq 1 \). (Hint: Show that \( x^{1/2}/x^{-1/2} = x \leq 1 \) for \( 0 < x \leq 1 \).)

(c) Show algebraically that
\[ x^{5/2} \geq x^{3/2} \]
for \( x \geq 1 \).

In Problems 53–56, sketch each scaling relation (Niklas, 1994).

53. In a sample based on 46 species, leaf area was found to be proportional to (stem diameter)^1.84. On the basis of your graph, as stem diameter increases, does leaf area increase or decrease?

54. In a sample based on 28 species, the volume fraction of spongy mesophyll was found to be proportional to (leaf thickness)^-0.49. (The spongy mesophyll is part of the internal tissue of a leaf blade.) On the basis of your graph, as leaf thickness increases, does the volume fraction of spongy mesophyll increase or decrease?

55. In a sample of 60 species of trees, wood density was found to be proportional to (breaking strength)^0.82. On the basis of your graph, does breaking strength increase as wood density increases? or as wood density decreases?

56. Suppose that a cube of length \( L \) and volume \( V \) has mass \( M \) and that \( M = 0.35V \). How does the length of the cube depend on its mass?

\[ \text{1.2.5} \]

57. Assume that a population size at time \( t \) is \( N(t) \) and that
\[ N(t) = 2^t, \quad t \geq 0 \]

(a) Find the population size for \( t = 0, 1, 2, 3, \) and \( 4 \).

(b) Graph \( N(t) \) for \( t \geq 0 \).

58. Assume that a population size at time \( t \) is \( N(t) \) and that
\[ N(t) = 40 \cdot 2^t, \quad t \geq 0 \]

(a) Find the population size at time \( t = 0 \).

(b) Show that
\[ N(t) = 40 \cdot 2^t, \quad t \geq 0 \]

(c) How long will it take until the population size reaches 1000? [Hint: Find \( t \) so that \( N(t) = 1000 \).]

59. The half-life of C14 is 5730 years. If a sample of C14 has a mass of 20 micrograms at time \( t \) = 0, how much is left after 2000 years?

60. The half-life of C14 is 5730 years. If a sample of C14 has a mass of 20 micrograms at time 0, how long will it take until (a) 10 grams and (b) 5 grams are left?

61. After 7 days, a particular radioactive substance decays to half of its original amount. Find the decay rate of this substance.

62. After 5 days, a particular radioactive substance decays to 37% of its original amount. Find the half-life of this substance.

63. Polonium 210 (Po210) has a half-life of 140 days.

(a) If a sample of Po210 has a mass of 300 micrograms, find a formula for the mass after \( t \) days.

(b) How long would it take for this sample to decay to 20% of its original amount?

(c) Sketch the graph of the amount of mass left after \( t \) days.

64. The half-life of C14 is 5730 years. Suppose that wood found at an archeological excavation site contains about 35% as much C14 (in relation to C12) as does living plant material. Determine when the wood was cut.

65. The half-life of C14 is 5730 years. Suppose that wood found at an archeological excavation site is 15,000 years old. How much C14 (based on C12 content) does the wood contain relative to living plant material?

66. The age of rocks of volcanic origin can be estimated with isotopes of argon 40 (Ar40) and potassium 40 (K40). K40 decays into Ar40 over time. If a mineral that contains potassium is buried under the right circumstances, argon forms and is trapped. Since argon is driven off when the mineral is heated to very high temperatures, rocks of volcanic origin do not contain argon when they are formed. The amount of argon found in such rocks can therefore be used to determine the age of the rock. Assume that a sample of volcanic rock contains 0.00047% K40. The sample also contains 0.000079% Ar40. How old is the rock? (The decay rate of K40 to Ar40 is 5.335 x 10^-10/yr.)

67. (Adapted from Moss, 1980.) Hall (1964) investigated the change in population size of the zooplankton species Daphnia galeata mendona in Base Line Lake, Michigan. The population size \( N(t) \) at time \( t \) was modeled by the equation
\[ N(t) = N_0e^{rt} \]
where \( N_0 \) denotes the population size at time 0. The constant \( r \) is called the intrinsic rate of growth.

(a) Plot \( N(t) \) as a function of \( t \) if \( N_0 = 100 \) and \( r = 2 \). Compare your graph against the graph of \( N(t) \) when \( N_0 = 100 \) and \( r = 3 \). Which population grows faster?
(b) The constant \( r \) is an important quantity because it describes how quickly the population changes. Suppose that you determine the size of the population at the beginning and at the end of a period of length 1, and you find that at the beginning there were 200 individuals and after one unit of time there were 250 individuals. Determine \( r \). [Hint: Consider the ratio \( N(t+1)/N(t) \).]

68. Fish are indeterminate growers; that is, they grow throughout their lifetime. The growth of fish can be described by the von Bertalanffy function

\[
L(x) = L_\infty(1 - e^{-kx})
\]

for \( x \geq 0 \), where \( L(x) \) is the length of the fish at age \( x \) and \( k \) and \( L_\infty \) are positive constants.

(a) Use a graphing calculator to graph \( L(x) \) for \( L_\infty = 20 \), for (i) \( k = 1 \) and (ii) \( k = 0.1 \).

(b) For \( k = 1 \), find \( x \) so that the length is 90% of \( L_\infty \). Repeat for 99% of \( L_\infty \). Can the fish ever attain length \( L_\infty \)? Interpret the meaning of \( L_\infty \).

(c) Compare the graphs obtained in (a). Which growth curve reaches 90% of \( L_\infty \) faster? Can you explain what happens to the curve of \( L(x) \) when you vary \( k \) (for fixed \( L_\infty \))?

\[ \text{1.2.6} \]

69. Which of the following functions is one to one (use the horizontal line test)?

(a) \( f(x) = x^2, x \geq 0 \)  
(b) \( f(x) = x^2, x \in \mathbb{R} \)  
(c) \( f(x) = \frac{1}{x}, x > 0 \)  
(d) \( f(x) = e^x, x \in \mathbb{R} \)  
(e) \( f(x) = \frac{1}{x}, x \neq 0 \)  
(f) \( f(x) = \frac{1}{x}, x > 0 \)

70. (a) Show that \( f(x) = x^3 - 1, x \in \mathbb{R} \), is one to one, and find its inverse together with its domain.

(b) Graph \( f(x) \) and \( f^{-1}(x) \) in one coordinate system, together with the line \( y = x \), and convince yourself that the graph of \( f^{-1}(x) \) can be obtained by reflecting the graph of \( f(x) \) about the line \( y = x \).

71. (a) Show that \( f(x) = x^2 + 1, x \geq 0 \), is one to one, and find its inverse together with its domain.

(b) Graph \( f(x) \) and \( f^{-1}(x) \) in one coordinate system, together with the line \( y = x \), and convince yourself that the graph of \( f^{-1}(x) \) can be obtained by reflecting the graph of \( f(x) \) about the line \( y = x \).

72. (a) Show that \( f(x) = \sqrt{x}, x \geq 0 \), is one to one, and find its inverse together with its domain.

(b) Graph \( f(x) \) and \( f^{-1}(x) \) in one coordinate system, together with the line \( y = x \), and convince yourself that the graph of \( f^{-1}(x) \) can be obtained by reflecting the graph of \( f(x) \) about the line \( y = x \).

73. (a) Show that \( f(x) = 1/x^3, x > 0 \), is one to one, and find its inverse together with its domain.

(b) Graph \( f(x) \) and \( f^{-1}(x) \) in one coordinate system, together with the line \( y = x \), and convince yourself that the graph of \( f^{-1}(x) \) can be obtained by reflecting the graph of \( f(x) \) about the line \( y = x \).

(b) The reciprocal of a function \( f(x) \) can be written as either \( 1/f(x) \) or \( [f(x)]^{-1} \). The point of this problem is to make clear that a reciprocal of a function has nothing to do with the inverse of a function. As an example, let \( f(x) = 2x + 1, x \in \mathbb{R} \). Find both \( [f(x)]^{-1} \) and \( f^{-1}(x) \), and compare the two functions. Graph all three functions together.

74. Find the inverse of \( f(x) = 3^x, x \in \mathbb{R} \), together with its domain, and graph both functions in the same coordinate system.

75. Find the inverse of \( f(x) = 5^x, x \in \mathbb{R} \), together with its domain, and graph both functions in the same coordinate system.

76. Find the inverse of \( f(x) = (1/2)^x, x \in \mathbb{R} \), together with its domain, and graph both functions in the same coordinate system.

77. Find the inverse of \( f(x) = (1/2)^x, x \in \mathbb{R} \), together with its domain, and graph both functions in the same coordinate system.

78. Find the inverse of \( f(x) = 2^x, x \geq 0 \), together with its domain, and graph both functions in the same coordinate system.

79. Find the inverse of \( f(x) = (1/3)^x, x \geq 0 \), together with its domain, and graph both functions in the same coordinate system.

80. Find the inverse of \( f(x) = (1/4)^x, x \geq 0 \), together with its domain, and graph both functions in the same coordinate system.

81. Simplify the following expressions:

(a) \( 2^{\log_2 x} \)  
(b) \( 3^{\log_3 x} \)  
(c) \( 5^{\log_{10} x} \)  
(d) \( 4^{-2\log_2 x} \)  
(e) \( 2^{\log_{10} x} \)  
(f) \( 4^{-\log_2 x} \)

82. Simplify the following expressions:

(a) \( \log_{10} 10 \)  
(b) \( \log_3 1 \)  
(c) \( \log_{10} 100 \)  
(d) \( \log_7 7 \)  
(e) \( \log_{10} 1 \)  
(f) \( \log_{10} 10 \)

83. Simplify the following expressions:

(a) \( \ln x + \ln x^{-2} \)  
(b) \( \ln x^2 - \ln x^{-1} \)  
(c) \( \ln(x^2 - 1) - \ln(x + 1) \)  
(d) \( \ln x + \ln x^{-2} \)

84. Simplify the following expressions:

(a) \( e^{\ln x} \)  
(b) \( e^{\ln x} \)  
(c) \( e^{2\ln x} \)  
(d) \( e^{2\ln x} \)

85. Write the following expressions in terms of base \( e \), and simplify:

(a) \( 3^x \)  
(b) \( 4^{x-1} \)  
(c) \( 2^{x-1} \)  
(d) \( 3^{-4x+1} \)

86. Write the following expressions in terms of base \( e \):

(a) \( \log_2 (x^2 - 1) \)  
(b) \( \log_5 (5x + 1) \)  
(c) \( \log_3 (x + 2) \)  
(d) \( \log_5 (2x^2 - 1) \)

87. Show that the function \( y = (1/2)^x \) can be written in the form \( y = e^{-\mu x} \), where \( \mu \) is a positive constant. Determine \( \mu \).

88. Show that if \( 0 < a < 1 \), then the function \( y = a^x \) can be written in the form \( y = e^{-\mu x} \), where \( \mu \) is a positive constant. Write \( \mu \) in terms of \( a \).

89. Assume that two DNA sequences of common origin, each of length 300 nucleotides, differ from each other by 47 nucleotides. Use the Jukes and Cantor correction of Example 15 to find an estimate for the number \( K \) of substitutions per site.

90. A community measure that takes both species abundance and species richness into account is the Shannon diversity index \( H \). To calculate \( H \), the proportion \( p_i \) of species \( i \) in the community is used. Assume that the community consists of \( S \) species. Then

\[
H = -\sum p_i \ln p_i + p_2 \ln p_2 + \cdots + p_S \ln p_S
\]

(a) Assume that \( S = 5 \) and that all species are equally abundant; that is, \( p_1 = p_2 = \cdots = p_5 \). Compute \( H \).

(b) Assume that \( S = 10 \) and that all species are equally abundant; that is, \( p_1 = p_2 = \cdots = p_{10} \). Compute \( H \).

(c) A measure of equitability (or evenness) of the species distribution can be measured by dividing the diversity index \( H \) by \( \ln S \). Compute \( H/\ln S \) for \( S = 5 \) and \( S = 10 \).

(d) Show that, in general, if there are \( N \) species and all species are equally abundant, then

\[
H/\ln S = 1
\]
In Problems 91–96, for each given pair of functions, use a graphing calculator to compare the functions. Describe what you see.

91. \( y = \sin x \) and \( y = 2\sin x \)
92. \( y = \sin x \) and \( y = \sin(2x) \)
93. \( y = \cos x \) and \( y = 2\cos x \)
94. \( y = \cos x \) and \( y = \cos(2x) \)
95. \( y = \tan x \) and \( y = 2\tan x \)
96. \( y = \tan x \) and \( y = \tan(2x) \)

97. Let \( f(x) = 3\sin(4x), \quad x \in \mathbb{R} \)
Find the amplitude and the period of \( f(x) \).

98. Let \( f(x) = -2\sin\left(\frac{x}{2}\right), \quad x \in \mathbb{R} \)
Find the amplitude and the period of \( f(x) \).

99. Let \( f(x) = 4\sin(2\pi x), \quad x \in \mathbb{R} \)
Find the amplitude and the period of \( f(x) \).

100. Let \( f(x) = -\frac{3}{2}\sin\left(\frac{\pi}{3}x\right), \quad x \in \mathbb{R} \)
Find the amplitude and the period of \( f(x) \).

In the preceding section, we introduced the functions most important to our study. You must be able to graph the following functions without a calculator: \( y = c, x, x^2, x^3, 1/x, e^x, \ln x, \sin x, \cos x, \sec x, \) and \( \tan x \). This will help you to sketch functions quickly and to come up with an analytical description of a function based on a graph. In this section, you will learn how to obtain new functions from these basic functions and how to graph them. In addition, we will introduce important transformations that are often used to display data graphically.

### 1.3.1 Graphing and Basic Transformations of Functions

In this subsection, we will recall some basic transformations: vertical and horizontal translations, reflections about \( x = 0 \) and \( y = 0 \), and stretching and compressing.

**Definition** The graph of \( y = f(x) + a \)

is a **vertical translation** of the graph of \( y = f(x) \). If \( a > 0 \), the graph of \( y = f(x) + a \) is shifted up \( a \) units; if \( a < 0 \), the graph of \( y = f(x) + a \) is shifted down \( |a| \) units.

This definition is illustrated in Figure 1.41, where we display \( y = x^2, y = x^2 + 2, \) and \( y = x^2 - 2 \).

**Definition** The graph of \( y = f(x - c) \)

is a **horizontal translation** of the graph of \( y = f(x) \). If \( c > 0 \), the graph of \( y = f(x) - c \) is shifted \( c \) units to the right; if \( c < 0 \), the graph of \( y = f(x) - c \) is shifted \( |c| \) units to the left.