1.1 Preliminaries

This section reviews some of the concepts and techniques from algebra and trigonometry that are frequently used in calculus. The problems at the end of the section will help you reacquaint yourself with this material.

1.1.1 The Real Numbers

The real numbers can most easily be visualized on the real-number line (see Figure 1.1), on which numbers are ordered so that if \( a < b \), then \( a \) is to the left of \( b \). Sets (collections) of real numbers are typically denoted by the capital letters \( A, B, C \), etc. To describe the set \( A \), we write

\[ A = \{ x : \text{condition} \} \]

where "condition" tells us which numbers are in the set \( A \). The most important sets in calculus are intervals. We use the following notations: If \( a < b \), then

the open interval \( (a, b) = \{ x : a < x < b \} \)

and

the closed interval \( [a, b] = \{ x : a \leq x \leq b \} \)

We also use half-open intervals:

\[ [a, b) = \{ x : a \leq x < b \} \quad \text{and} \quad (a, b] = \{ x : a < x \leq b \} \]

Unbounded intervals are sets of the form \( \{ x : x > a \} \). Here are the possible cases:

\[ [a, \infty) = \{ x : x \geq a \} \]

\[ (-\infty, a] = \{ x : x \leq a \} \]

\[ (a, \infty) = \{ x : x > a \} \]

\[ (-\infty, a) = \{ x : x < a \} \]

The symbols "\( \infty \)" and "\( -\infty \)" mean "plus infinity" and "minus infinity," respectively. These symbols are not real numbers, but are used merely for notational convenience. The real-number line, denoted by \( \mathbb{R} \), does not have endpoints, and we can write \( \mathbb{R} \) in the following equivalent forms:

\[ \mathbb{R} = \{ x : -\infty < x < \infty \} = (-\infty, \infty) \]

The location of the number 0 on the real-number line is called the origin, and we can measure the distance of the number \( x \) to the origin. For instance, \( -5 \) is 5 units to the left of the origin. A convenient notation for measuring distances from the origin on the real-number line is the absolute value of a real number.

**Definition** The absolute value of a real number \( a \), denoted by \( |a| \), is

\[ |a| = \begin{cases} 
  a & \text{if } a \geq 0 \\
  -a & \text{if } a < 0 
\end{cases} \]

For example, \( |-7| = -(-7) = 7 \). We can use absolute values to find the distance between any two numbers \( x_1 \) and \( x_2 \) as follows:

\[ \text{distance between } x_1 \text{ and } x_2 = |x_1 - x_2| \]

Note that \( |x_1 - x_2| = |x_2 - x_1| \). To find the distance between \(-2\) and \(4\), we compute \( |-2 - 4| = |-6| = 6 \), or \( |4 - (-2)| = |6| = 6 \).
We will frequently need to solve equations containing absolute values, for which the following property is useful:

Let \( b \geq 0 \). Then

1. For \( a \geq 0 \), \(|a| = b\) is equivalent to \( a = b \).
2. For \( a < 0 \), \(|a| = b\) is equivalent to \(-a = b\).

**Example 1**

Solve \(|x - 4| = 2\).

If \( x - 4 \geq 0 \), then \( x - 4 = 2 \) and thus \( x = 6 \). If \( x - 4 < 0 \), then \(- (x - 4) = 2\) and thus \( x = 2 \). The solutions, illustrated graphically in Figure 1.2, are therefore \( x = 6 \) and \( x = 2 \). The points of intersection of \( y = |x - 4|\) and \( y = 2\) are at \( x = 6\) and \( x = 2\). Solving \(|x - 4| = 2\) can also be interpreted as finding the two numbers that have distance 2 from 4.

![Graph](image)

*Figure 1.2* The graph of \( y = |x - 4|\) and \( y = 2\). The points of intersection are at \( x = 6 \) and \( x = 2 \).

We write the solution of an equation of the form \(|a| = |b|\) as either \( a = b \) or \( a = -b \), illustrated in the next example.

**Example 2**

Solve \(|\frac{3}{2}x - 1| = |\frac{1}{2}x + 1|\).

Solution

Either

\[
\begin{align*}
\frac{3}{2}x - 1 &= \frac{1}{2}x + 1 \\
\frac{3}{2}x - 1 &= -\left(\frac{1}{2}x + 1\right)
\end{align*}
\]

\[
\begin{align*}
x &= 2 \\
2x &= 0 \\
x &= 0
\end{align*}
\]

A graphical solution of this example is shown in Figure 1.3.

Returning to Example 1, where we found the two points whose distance from 4 was equal to 2, we can also try to find those points whose distance from 4 is less than (or greater than) 2. This amounts to solving inequalities with absolute values. Looking back at Figure 1.2, we see that the set of x-values whose distance from 4 is less than 2 (i.e., \(|x - 4| < 2\)) is the interval \((2, 6)\). Similarly, the set of x-values whose distance from 4 is greater than 2 (i.e., \(|x - 4| > 2\)) is the union of the two intervals \((-\infty, 2)\) and \((6, \infty)\), or \((-\infty, 2) \cup (6, \infty)\).
Figure 1.3 The graphs of \( y = |\frac{1}{2}x - 1| \) and \( y = |\frac{1}{2}x + 1| \).
The points of intersection are at \( x = 0 \) and \( x = 2 \).

In general, to solve absolute-value inequalities, the following two properties are useful:

Let \( b > 0 \). Then

1. \(|a| < b \) is equivalent to \(-b < a < b\).
2. \(|a| > b \) is equivalent to \( a > b \) or \( a < -b \).

**Example 3**

**Solution**

(a) Solve \(|2x - 5| < 3\).
(b) Solve \(|4 - 3x| \geq 2\).

(a) We rewrite \(|2x - 5| < 3\) as

\[-3 < 2x - 5 < 3\]

Adding 5 to all three parts, we obtain

\[2 < 2x < 8\]

Dividing the result by 2, we find that

\[1 < x < 4\]

The solution is therefore the set \( \{x : 1 < x < 4\} \). In interval notation, the solution can be written as the open interval \((1, 4)\).

(b) To solve \(|4 - 3x| \geq 2\), we go through the following steps:

\[4 - 3x \geq 2\]
\[-3x \geq -6\]
\[x \leq 2\]

or

\[4 - 3x \leq -2\]
\[-3x \leq -6\]
\[x \geq 2\]

The solution is the set \( \{x : x \geq 2 \text{ or } x \leq \frac{2}{3}\} \), or, in interval notation, \((-\infty, \frac{2}{3}] \cup [2, \infty)\).

1.1.2 Lines in the Plane

We will frequently encounter situations in which the relationship between quantities can be described by a linear equation. For example, when a weight is attached to a helical spring made of some elastic material (and the weight is not too heavy), the relationship between the length \( y \) of the spring and the weight \( x \) is

\[ y = y_0 + kx \tag{1.1} \]

where \( y_0 \) denotes the length of the spring when no weight is attached to it and \( k \) is a positive constant. Equation (1.1) is an example of a linear equation, and we say that \( x \) and \( y \) satisfy a linear equation.
The standard form of a linear equation is given by

$$Ax + By + C = 0$$

where $A$, $B$, and $C$ are constants, $A$ and $B$ are not both equal to 0, and $x$ and $y$ are the two variables. In algebra, you learned that the graph of a linear equation is a straight line.

If the two points $(x_1, y_1)$ and $(x_2, y_2)$ lie on a straight line, then the slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

(See Figure 1.4.) Two points (or one point and the slope) are sufficient to determine the equation of a straight line.

If you are given one point and the slope, you can use the point-slope form of a straight line to write its equation, given by

$$y - y_0 = m(x - x_0)$$

where $m$ is the slope and $(x_0, y_0)$ is a point on the line. If you are given two points, first compute the slope and then use one of the points and the slope to find the equation of the straight line in point-slope form.

Lastly, the most frequently used form of a linear equation is the slope-intercept form

$$y = mx + b$$

where $m$ is the slope and $b$ is the $y$-intercept, which is the point of intersection of the line with the $y$-axis; the $y$-intercept has coordinates $(0, b)$.

We summarize these three forms of linear equations in the following box:

$$\begin{align*}
Ax + By + C &= 0 \quad \text{(Standard Form)} \\
y - y_0 &= m(x - x_0) \quad \text{(Point-Slope Form)} \\
y &= mx + b \quad \text{(Slope-Intercept Form)}
\end{align*}$$

**EXAMPLE 4**

Determine, in slope-intercept form, the equation of the line passing through $(-2, 1)$ and $(3, -\frac{1}{2})$.

**Solution**

The slope of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-\frac{1}{2} - 1}{3 - (-2)} = \frac{-\frac{3}{2}}{5} = -\frac{3}{10}$$

Using the point-slope form with $(-2, 1)$, we find that

$$y - 1 = -\frac{3}{10} (x - (-2))$$

or, in slope-intercept form,

$$y = -\frac{3}{10} x + \frac{2}{5}$$

We could have used the other point, $(3, -\frac{1}{2})$, and obtained the same result.

We now recall two special cases that we illustrate in Figure 1.5:

- $y = k$ horizontal line (slope 0)
- $x = h$ vertical line (slope undefined)
In the next example, we show how to determine the slope and the y-intercept of a given straight line.

**EXAMPLE 5**

**Solution**

Determine the slope and the y-intercept of the line \(3y - 2x + 9 = 0\).

We solve for \(y\) in \(3y = 2x - 9\). We obtain \(y = \frac{2}{3}x - 3\). We can now read off the slope \(m = \frac{2}{3}\) and the y-intercept \(b = -3\).

When two quantities \(x\) and \(y\) are linearly related so that

\[ y = mx \]

we say that \(y\) is **proportional** to \(x\), with \(m\) denoting the **constant of proportionality**, and we write

\[ y \propto x \]

The symbol \(\propto\) is read “is proportional to.” If we write Equation (1.1) in the form

\[ y - y_0 = kx \]

then the change in length \(y - y_0\) is proportional to the attached weight with constant of proportionality \(k\), and we can write

\[ y - y_0 \propto x \]

There are two more properties of straight lines we wish to mention. When two lines \(l_1\) and \(l_2\) in the plane have no points in common or are identical, they are called **parallel**, denoted by \(l_1 \parallel l_2\). The following criterion is useful in deciding whether two lines are parallel: Two noncoincident lines \(l_1\) and \(l_2\) are parallel \((l_1 \parallel l_2)\) if and only if their slopes are identical. For two noncoincident, nonvertical lines \(l_1\) and \(l_2\) with slopes \(m_1\) and \(m_2\), respectively, the criterion becomes

\[ l_1 \parallel l_2 \quad \text{if and only if} \quad m_1 = m_2 \]

Two lines \(l_1\) and \(l_2\) are called **perpendicular** \((l_1 \perp l_2)\) if their intersection forms an angle of 90°. The following criterion is useful for deciding whether two lines are perpendicular: Two nonvertical lines are perpendicular if and only if their slopes are negative reciprocals. That is, if \(l_1\) and \(l_2\) are nonvertical lines with slopes \(m_1\) and \(m_2\), then

\[ l_1 \perp l_2 \quad \text{if and only if} \quad m_1m_2 = -1 \]

We will prove this result in Problem 54 at the end of this section.

**1.1.3 Equation of the Circle**

A **circle** is the set of all points at a given distance, called the **radius**, from a given point, called the **center**. If \(r\) is the distance from \((x_0, y_0)\) to \((x, y)\) (see Figure 1.6), then, using the Pythagorean theorem, we find that

\[ r^2 = (x - x_0)^2 + (y - y_0)^2 \]

If \(r = 1\) and \((x_0, y_0) = (0, 0)\), the circle is called the **unit circle**.

**EXAMPLE 6**

**Solution**

Find the equation of the circle with center \((2, 3)\) and passing through \((5, 7)\).

Using the Pythagorean theorem, we can compute the distance in the plane between \((2, 3)\) and \((5, 7)\):

\[ \sqrt{(5 - 2)^2 + (7 - 3)^2} = \sqrt{9 + 16} = 5 \]

Thus, this circle has radius 5 and center \((2, 3)\), and its equation is

\[ 25 = (x - 2)^2 + (y - 3)^2 \]
1.1.4 Trigonometry

We will need a few results from trigonometry. Recall that angles are measured in either degrees or radians and that a complete revolution on a unit circle (Figure 1.7) corresponds to 360°, or 2π. For reasons that will become clear, the radian measure is preferred in calculus. To convert between degree and radian measure, we use the formula

\[ \frac{\theta \text{ measured in degrees}}{360°} = \frac{\theta \text{ measured in radians}}{2\pi} \]

For instance, to convert 23° into radian measure, we compute

\[ \theta = \frac{23° \times 2\pi}{360°} = 0.401 \]

To convert \( \frac{\pi}{6} \) into degrees, we compute

\[ \theta = \frac{\pi}{6} \times \frac{360°}{2\pi} = 30° \]

There are four trigonometric functions that you should be familiar with: sine, cosine, tangent, and secant; the other two, cotangent and cosecant, are rarely used. The six are defined on a unit circle (see Figure 1.7) and are abbreviated as sin, cos, tan, sec, cot, and csc, respectively. Recall that a positive angle is measured counterclockwise from the positive x-axis, whereas a negative angle is measured clockwise. The six trigonometric functions are defined as follows:

\[
\begin{align*}
\sin \theta &= y = \frac{y}{1} \\
\cos \theta &= x = \frac{x}{1} \\
\tan \theta &= \frac{y}{x} \\
\sec \theta &= \frac{1}{\cos \theta} = \frac{x}{1} \\
\csc \theta &= \frac{1}{\sin \theta} = \frac{y}{1} \\
\cot \theta &= \frac{1}{\tan \theta} = \frac{x}{y}
\end{align*}
\]

There are a number of frequently used trigonometric identities. First, since \( \tan \theta = \frac{y}{x} \) with \( y = \sin \theta \) and \( x = \cos \theta \), it follows that

\[ \tan \theta = \frac{\sin \theta}{\cos \theta} \]

Now, applying the Pythagorean theorem to the triangle in Figure 1.7 and using the notation \( \sin^2 \theta = (\sin \theta)^2 \), we find that

\[ \sin^2 \theta + \cos^2 \theta = 1 \]

Next, if we divide the preceding identity by \( \cos^2 \theta \), we obtain

\[ \frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \frac{1}{\cos^2 \theta} \]

Using \( \tan \theta = \frac{\sin \theta}{\cos \theta} \) and \( \sec \theta = \frac{1}{\cos \theta} \), we can write this as

\[ \tan^2 \theta + 1 = \sec^2 \theta \]

In the next example, we solve a trigonometric equation.

**Example 7**

Solve

\[ 2 \sin \theta \cos \theta = \cos \theta \quad \text{on} \quad [0, 2\pi) \]
Solution

We should not be tempted to cancel $\cos \theta$ on each side; this would cause us to lose solutions. Instead, we bring $\cos \theta$ to the left side and factor $\cos \theta$ to obtain

$$\cos \theta (2 \sin \theta - 1) = 0$$

That is,

$$\cos \theta = 0 \quad \text{or} \quad 2 \sin \theta - 1 = 0$$

Solving $\cos \theta = 0$, we find that

$$\theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{3\pi}{2}$$

Solving $2 \sin \theta - 1 = 0$, we get

$$\sin \theta = \frac{1}{2}$$

which yields

$$\theta = \frac{\pi}{6} \quad \text{or} \quad \theta = \frac{5\pi}{6}$$

The solution set is therefore $\{\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\}$.

Figure 1.8 yields the following two identities when we compare the two angles $\theta$ and $-\theta$ (a positive angle is measured counterclockwise from the positive $x$-axis, whereas a negative angle is measured clockwise):

$$\sin(-\theta) = -\sin \theta \quad \text{and} \quad \cos(-\theta) = \cos \theta$$

Some exact trigonometric values are collected in Table 1-1. Of course, $\frac{1}{\sqrt{2}} = 0$, $\frac{1}{\sqrt{3}} = \frac{1}{3}$, and $\frac{1}{\sqrt{4}} = 1$, and you should memorize these simplified values. Rewriting Table 1-1 will make it easier to re-create the table in case you forget the exact values. Using $\tan \theta = \sin \theta / \cos \theta$, you immediately get the values for $\tan \theta$.

<table>
<thead>
<tr>
<th>Angle $\theta$</th>
<th>$0$</th>
<th>$\frac{\pi}{6}$</th>
<th>$\frac{\pi}{4}$</th>
<th>$\frac{\pi}{3}$</th>
<th>$\frac{\pi}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0^\circ)$</td>
<td></td>
<td>$(30^\circ)$</td>
<td>$(45^\circ)$</td>
<td>$(60^\circ)$</td>
<td>$(90^\circ)$</td>
</tr>
<tr>
<td>$\sin \theta$</td>
<td>$\frac{1}{2} \sqrt{0}$</td>
<td>$\frac{1}{2} \sqrt{1}$</td>
<td>$\frac{1}{2} \sqrt{2}$</td>
<td>$\frac{1}{2} \sqrt{3}$</td>
<td>$\frac{1}{2} \sqrt{4}$</td>
</tr>
<tr>
<td>$\cos \theta$</td>
<td>$\frac{1}{2} \sqrt{4}$</td>
<td>$\frac{1}{2} \sqrt{3}$</td>
<td>$\frac{1}{2} \sqrt{2}$</td>
<td>$\frac{1}{2} \sqrt{1}$</td>
<td>$\frac{1}{2} \sqrt{0}$</td>
</tr>
</tbody>
</table>

1.1.5 Exponentials and Logarithms

Exponentials and logarithms are particularly important in biological contexts. An exponential is an expression of the form

$$a^r$$

where $a$ is called the base and $r$ the exponent. Unless $r$ is an integer or unless $r$ is a rational number of the form $p/q$ where $p$ is an integer and $q$ is an odd integer, we will assume that $a$ is positive. We summarize some of the properties of an exponential as follows:

$$a^r a^s = a^{r+s} \quad \text{and} \quad (ab)^r = a^r b^r$$

$$\frac{a^r}{a^s} = a^{r-s} \quad \frac{a^r}{b^r} = \left(\frac{a}{b}\right)^r$$

$$a^{-r} = \frac{1}{a^r} \quad (a^r)^s = a^{rs}$$
EXAMPLE 8
Evaluate the following exponential expressions:
(a) \(3^2 \cdot 3^{5/2} = 3^{2 + 5/2} = 3^{9/2}\)
(b) \(2^{-4} \cdot 2^3 = \frac{2^{-1}}{2^3} = 2^{-1-2} = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}\)
(c) \(\frac{a^k \cdot a^{3k}}{a^{5k}} = a^{k+3k-5k} = a^{-k} = \frac{1}{a^k}\)

Logarithms allow us to solve equations of the form
\[2^x = 8\]
The solution of this equation is \(x = 3\), which we can write as
\[x = \log_2 8 = 3\]
In other words, a logarithm is an exponent. The expression
\[\log_a y\]
is the exponent on the base \(a\) that yields the number \(y\). Logarithms are defined only for \(y > 0\) (where the base is assumed to be positive and different from 1). We have the following correspondence between logarithms and exponentials:
\[x = \log_a y \quad \text{is equivalent to} \quad y = a^x\]

EXAMPLE 9
Which real number \(x\) satisfies
(a) \(\log_3 x = -2\)  \hspace{1cm} (b) \(\log_{1/2} 8 = x^2\)

**Solution**
(a) We write this in the equivalent form
\[x = 3^{-2}\]
Hence,
\[x = \frac{1}{3^2} = \frac{1}{9}\]
(b) We write this in the equivalent form
\[\left(\frac{1}{2}\right)^x = 8\]
\[2^{-x} = 2^3\]
\[2^x = 2^{-3}\]
Setting the exponents equal to each other, we find that \(x = -3\). Note that, in order to compare exponents, the bases must be the same.

Some important properties of logarithms are as follows:
\[\log_a (xy) = \log_a x + \log_a y\]
\[\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y\]
\[\log_a x^r = r \log_a x\]

The most important logarithm is the **natural logarithm**, which has the number \(e\) as its base. The number \(e\) is an irrational number whose value is approximately 2.7182818. The natural logarithm is written \(\ln x\); that is, \(\log_e x = \ln x\).
Assume that \( x \) and \( y \) are positive, and simplify the following expressions:

(a) \( \log_3(9x^2) = \log_3 9 + \log_3 x^2 = 2 + 2 \log_3 x \)

(b) \( \log_5 \frac{x^2 + 3}{x} = \log_5(x^2 + 3) - \log_5 x = \log_5(x^2 + 3) - 1 - \log_5 x \)

[Note that \( \log_5(x^2 + 3) \) cannot be simplified any further.]

(c) \( -\ln \frac{1}{2} = \ln (\frac{1}{2})^{-1} = \ln 2 \)

(d) \( \ln \frac{3x^2}{\sqrt{y}} = \ln 3 + \ln x^2 - \ln \sqrt{y} = \ln 3 + 2 \ln x - \frac{1}{2} \ln y \)

(In the last step, we used the fact that \( \sqrt{y} = y^{1/2} \).

In algebra, you learned how to solve equations of the form \( e^{2x} = 3 \) or \( \ln(x + 1) = 5 \). We will need to do this frequently. The key to solving such equations are the two identities

\[
\log_a a^x = x \quad \text{and} \quad a^{\log_a x} = x
\]

The next example illustrates how to use these identities.

**EXAMPLE 11**

Solve for \( x \).

(a) \( e^{2x} = 3 \)  
(b) \( \ln(x + 1) = 5 \)  
(c) \( 5^{2x-1} = 2^x \)

**Solution**

(a) To solve \( e^{2x} = 3 \) for \( x \), we take logarithms to base \( e \) on both sides:

\[
\ln e^{2x} = \ln 3
\]

But \( \ln e^{2x} = 2x \); hence,

\[
2x = \ln 3, \quad \text{or} \quad x = \frac{\ln 3}{2}
\]

(b) To solve \( \ln(x + 1) = 5 \), we write the equation in exponential form:

\[
e^{\ln(x+1)} = e^5
\]

This simplifies to

\[
x + 1 = e^5, \quad \text{or} \quad x = e^5 - 1
\]

(c) To solve \( 5^{2x-1} = 2^x \) for \( x \), we observe that the two bases are different. We therefore cannot compare the exponents directly. Instead, we take logarithms on both sides. Any positive base (different from 1) for the logarithm would work, and we choose base \( e \), since it is the most commonly used base in calculus. Doing so yields

\[
\ln 5^{2x-1} = \ln 2^x
\]

or, after simplifying,

\[
(2x - 1) \ln 5 = x \ln 2
\]

Solving for \( x \), we find that

\[
2x \ln 5 - x \ln 2 = \ln 5
\]

\[
x (2 \ln 5 - \ln 2) = \ln 5
\]

Hence,

\[
x = \frac{\ln 5}{2 \ln 5 - \ln 2}
\]
1.1.6 Complex Numbers and Quadratic Equations

The square of a real number is always nonnegative. However, there are situations in which we need to take a square root of a negative number. Since the resulting square root cannot be a real number, we introduce a new symbol, which we denote by \( i \), that will allow us to deal with this case. We set

\[
i^2 = -1
\]

The symbol \( i \) is called the **imaginary unit**. Thus, instead of writing \( \sqrt{-17} \), for instance, we can now write \( i\sqrt{17} \).

The symbol \( i \) allows us to introduce a new number system, the set of **complex numbers**:

A complex number is a number of the form

\[
z = a + bi
\]

where \( a \) and \( b \) are real numbers. The real number \( a \) is the **real part** of \( a + bi \), and the real number \( b \) is the **imaginary part**.

For instance, \(-3 + 7i\) has real part \(-3\) and imaginary part \(7\), and \(2 - 5i\) has real part \(2\) and imaginary part \(-5\). Since \(a + 0i = a\), it follows that the set of real numbers is a subset of the set of complex numbers. Complex numbers of the form \( bi \) are called **purely imaginary numbers**.

Two complex numbers are equal if their respective real and imaginary parts are equal; that is,

\[
a + bi = c + di \quad \text{if and only if} \quad a = c \quad \text{and} \quad b = d
\]

To add two complex numbers, we use the following rule:

\[(a + bi) + (c + di) = (a + c) + (b + d)i\]

This rule says that real and imaginary parts are added separately. To calculate the product of two complex numbers, we proceed as follows:

\[
(a + bi)(c + di) = ac + adi + bci + bdi^2
\]

\[= ac + (ad + bc)i - bd\]

\[= (ac - bd) + (ad + bc)i
\]

Note that we used \( i^2 = -1 \) in the penultimate step. There is no need to memorize the product of two complex numbers, since we can always compute it by the distributive law.

**Example 12**

Find

(a) \((2 + 3i) - (5 - 6i)\), \quad (b) \((5 - 3i)(1 + 2i)\).

Solution

(a) \((2 + 3i) - (5 - 6i) = 2 + 3i - 5 + 6i = -3 + 9i\),

(b) \((5 - 3i)(1 + 2i) = 5 + 10i - 3i - 6i^2 = 5 + 7i - (6)(-1) = 11 + 7i\).

If \(z = a + bi\) is a complex number, its **conjugate**, denoted by \(\overline{z}\), is defined as

\[
\overline{z} = a - bi
\]
For complex numbers \( z \) and \( w \), it can be shown (see Problems 113–115) that

\[
\begin{align*}
\overline{(z)} &= z \\
\overline{z + w} &= \overline{z} + \overline{w} \\
\overline{zw} &= \overline{z} \overline{w}
\end{align*}
\]

Furthermore, if we multiply a complex number by its conjugate, we find that

\[
z\overline{z} = (a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2
\]

That is,

If \( z = a + bi \), then \( z\overline{z} = a^2 + b^2 \)

**EXAMPLE 13**

Let \( z = 3 + 2i \).

(a) Find \( \overline{z} \).

(b) Compute \( z\overline{z} \).

(a) \( \overline{z} = 3 - 2i \).

(b) \( z\overline{z} = (3 + 2i)(3 - 2i) = 9 - 4i^2 = 9 + 4 = 13 \).

We encounter complex numbers primarily when we solve quadratic equations. Recall that, to solve

\[ax^2 + bx + c = 0\]

for \( a \neq 0 \), we use the quadratic formula

\[x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]

where \( x_{1,2} \) refers to the two solutions \( x_1 \) (with the “+” sign) and \( x_2 \) (with the “−” sign).

**EXAMPLE 14**

Solve

\[x^2 + 4x + 5 = 0\]

Using the quadratic formula, we obtain

\[x_{1,2} = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2(1)} = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i\]

If we allowed solutions only in the real-number system, we would conclude that \( x^2 + 4x + 5 = 0 \) has no solutions. But if we allow solutions in the complex number system, we find that

\[x_{1,2} = \frac{-4 \pm \sqrt{4i^2}}{2} = \frac{-4 \pm 2i}{2} = \frac{2(-2 \pm i)}{2} = -2 \pm i\]

That is, \( x_1 = -2 + i \) and \( x_2 = -2 - i \).
The term $b^2 - 4ac$ under the square root sign in the quadratic formula is called the discriminant. If the discriminant is nonnegative, the two solutions of the corresponding quadratic equation are real. (When the discriminant is equal to 0, the two solutions are identical.) If the discriminant is negative, the two solutions are complex conjugates of each other.

**Example 15**

Without solving

$$2x^2 - 3x + 7 = 0$$

what can you say about the solution?

**Solution**

We compute the discriminant

$$b^2 - 4ac = (-3)^2 - (4)(2)(7) = 9 - 56 = -47 < 0$$

Since the discriminant is negative, the equation $2x^2 - 3x + 7 = 0$ has two complex solutions, which are conjugates of each other.

### Section 1.1 Problems

1. Find the two numbers that have distance 3 from -1 by (a) measuring the distances on the real-number line and (b) solving an appropriate equation involving an absolute value.

2. Find all pairwise distances between the numbers -5, 2, and 7 by (a) measuring the distances on the real-number line and (b) computing the distances by using absolute values.

3. Solve the following equations:
   (a) $|2x - 4| = 6$
   (b) $|x - 3| = 2$
   (c) $|2x + 3| = 5$
   (d) $|7 - 3x| = -2$

4. Solve the following equations:
   (a) $|2x + 4| = |5x - 2|$
   (b) $|5 - 3u| = |3 + 2u|$
   (c) $|4 + \frac{1}{x}| = \left|\frac{3}{2} - x\right|$
   (d) $|2x - 3| = |7 - x|$

5. Solve the following inequalities:
   (a) $|5x - 2| \leq 4$
   (b) $|1 - 3x| > 8$
   (c) $|7x + 4| \geq 3$
   (d) $|6 - 5x| < 7$

6. Solve the following inequalities:
   (a) $|2x + 3| < 6$
   (b) $|3 - 4x| \geq 2$
   (c) $|x + 5| \leq 1$
   (d) $|7 - 2x| < 0$

**1.1.2**

In Problems 7—42, determine the equation of the line that satisfies the stated requirements. Put the equation in standard form.

7. The line passing through (2, 4) with slope $-\frac{1}{3}$

8. The line passing through (1, -2) with slope 2

9. The line passing through (0, -2) with slope -3

10. The line passing through (-3, 5) with slope 1/2

11. The line passing through (-2, -3) and (1, 4)

12. The line passing through (1, -4) and (2, 1)

13. The line passing through (0, 4) and (3, 0)

14. The line passing through (1, -1) and (4, 5)

15. The horizontal line through (3, \frac{1}{2})

16. The horizontal line through (0, -1)

17. The vertical line through (-1, \frac{1}{2})

18. The vertical line through (2, -3)

19. The line with slope 3 and y-intercept (0, 2)

20. The line with slope -1 and y-intercept (0, -3)

21. The line with slope 1/2 and y-intercept (0, 2)

22. The line with slope -1/3 and y-intercept (0, -1)

23. The line with slope -2 and x-intercept (1, 0)

24. The line with slope 1 and x-intercept (-2, 0)

25. The line with slope -1/4 and x-intercept (3, 0)

26. The line with slope 1/5 and x-intercept (-1/2, 0)

27. The line passing through (2, -3) and parallel to $x + 2y - 4 = 0$

28. The line passing through (1, 2) and parallel to $x - 3y - 6 = 0$

29. The line passing through (-1, -1) and parallel to the line passing through (0, 1) and (3, 0)

30. The line passing through (2, -3) and parallel to the line passing through (0, -1) and (2, 1)

31. The line passing through (1, 4) and perpendicular to $2y - 5x + 7 = 0$

32. The line passing through (-1, 1) and perpendicular to $x - y + 3 = 0$

33. The line passing through (5, -1) and perpendicular to the line passing through (-2, 1) and (1, -2)

34. The line passing through (4, -1) and perpendicular to the line passing through (-2, 0) and (1, 1)

35. The line passing through (4, 2) and parallel to the horizontal line passing through (1, -2)

36. The line passing through (-1, 5) and parallel to the horizontal line passing through (2, -1)

37. The line passing through (-1, 1) and parallel to the vertical line passing through (2, -1)

38. The line passing through (3, 1) and parallel to the vertical line passing through (-1, -2)

39. The line passing through (1, -3) and perpendicular to the horizontal line passing through (-1, -1)
40. The line passing through (4, 2) and perpendicular to the horizontal line passing through (3, 1)
41. The line passing through (7, 3) and perpendicular to the vertical line passing through (−2, 4)
42. The line passing through (−2, 5) and perpendicular to the vertical line passing through (1, 4)

43. To convert a length measured in feet to a length measured in centimeters, we use the facts that a length measured in feet is proportional to a length measured in centimeters and that 1 ft corresponds to 30.5 cm. If x denotes the length measured in ft and y denotes the length measured in cm, then

\[ y = 30.5x \]

(a) Explain how to use this relationship.
(b) Use the relationship to convert the following measurements into centimeters:
   (i) 6 ft    (ii) 3 ft, 2 in   (iii) 1 ft, 7 in
(c) Use the relationship to convert the following measurements into ft:
   (i) 173 cm  (ii) 75 cm  (iii) 48 cm

44. (a) To convert the weight of an object from kilograms (kg) to pounds (lb), you use the facts that a weight measured in kilograms is proportional to a weight measured in pounds and that 1 kg corresponds to 2.20 lb. Find an equation that relates weight measured in kilograms to weight measured in pounds.
(b) Use your answer in (a) to convert the following measurements:
   (i) 63 lb   (ii) 150 lb  (iii) 2.5 kg  (iv) 140 kg

45. Assume that the distance a car travels is proportional to the time it takes to cover the distance. Find an equation that relates distance and time if it takes the car 15 min to travel 10 mi. What is the constant of proportionality if distance is measured in miles and time is measured in hours?

46. Assume that the number of seeds a plant produces is proportional to its above-ground biomass. Find an equation that relates number of seeds and above-ground biomass if a plant that weighs 217 g has 17 seeds.

47. Experimental study plots are often squares of length 1 m. If 1 ft corresponds to 0.305 m, compute the area of a square plot of length 1 m in ft².

48. Large areas are often measured in hectares (ha) or in acres. If 1 ha = 10,000 m² and 1 acre = 4046.86 m², how many acres is 1 hectare?

49. To convert the volume of a liquid measured in ounces to a volume measured in liters, we use the fact that 1 liter equals 33.81 ounces. Denote by x the volume measured in ounces and by y the volume measured in liters. Assume a linear relationship between these two units of measurements.
   (a) Find the equation relating x and y.
   (b) A typical soda can contains 12 ounces of liquid. How many liters is this?
50. To convert a distance measured in miles to a distance measured in kilometers, we use the fact that 1 mile equals 1.609 kilometers. Denote by x the distance measured in miles and by y the distance measured in kilometers. Assume a linear relationship between these two units of measurements.
   (a) Find an equation relating x and y.
   (b) The distance between Minneapolis and Madison is 261 miles. How many kilometers is this?

51. Car speed in many countries is measured in kilometers per hour. In the United States, car speed is measured in miles per hour. To convert between these units, use the fact that 1 mile equals 1.609 kilometers.
   (a) The speed limit on many U.S. highways is 55 miles per hour. Convert this number into kilometers per hour.
   (b) The recommended speed limit on German highways is 130 kilometers per hour. Convert this number into miles per hour.

52. (a) The Celsius scale is devised so that 0°C is the freezing point of water (at 1 atmosphere of pressure) and 100°C is the boiling point of water (at 1 atmosphere of pressure). If you are more familiar with the Fahrenheit scale, then you know that water freezes at 32°F and boils at 212°F. Find a linear equation that relates temperature measured in degrees Celsius to temperature measured in degrees Fahrenheit.
   (b) The normal body temperature in humans ranges from 97.6°F to 99.6°F. Convert this temperature range into degrees Celsius.

53. (a) The Kelvin (K) scale is an absolute scale of temperature. The zero point of the scale (0 K) denotes absolute zero, the coldest possible temperature: that is, no body can have a temperature below 0 K. It has been determined experimentally that 0 K corresponds to −273.15°C. If 1 K denotes the same temperature difference as 1°C, find an equation that relates the Kelvin and Celsius scales.
   (b) Pure nitrogen and pure oxygen can be produced cheaply by first liquefying purified air and then allowing the temperature of the liquid air to rise slowly. Since nitrogen and oxygen have different boiling points, they are distilled at different temperatures. The boiling point of nitrogen is 77.4 K and of oxygen is 90.2 K. Convert each of these boiling-point temperatures into Celsius. If you solved Problem 52(a), convert the boiling-point temperatures into Fahrenheit as well. Consider the two techniques described for distilling nitrogen and oxygen. Which element gets distilled first?

54. Use the following steps to show that if two non-vertical lines l₁ and l₂ with slopes m₁ and m₂, respectively, are perpendicular, then m₁m₂ = −1. Assume that m₁ < 0 and m₂ > 0.
   (a) Use a graph to show that if θ₁ and θ₂ are the respective angles of inclination of the lines l₁ and l₂, then θ₁ = θ₂ + π. (The angle of inclination of a line is the angle θ ∈ [0, π) between the line and the positively directed x-axis.)
   (b) Use the fact that \( \tan(\pi - x) = -\tan x \) to show that \( m_1 = \tan \theta_1 \) and \( m_2 = \tan \theta_2 \).
   (c) Use the fact that \( \tan(\frac{\pi}{2} - x) = \cot x \) and \( \cot(-x) = -\cot x \) to show that \( m_1 = -\cot \theta_1 \).
   (d) From the latter equation, deduce the truth of the claim set forth at the beginning of this problem.

55. Find the equation of a circle with center (−1, 4) and radius 3.
56. Find the equation of a circle with center (2, 3) and radius 4.
57. (a) Find the equation of a circle with center (2, 5) and radius 3.
   (b) Where does the circle intersect the y-axis?
   (c) Does the circle intersect the x-axis? Explain.
58. (a) Find all possible radii of a circle centered at (3, 6) so that the circle intersects only one axis.
Find all possible radii of a circle centered at (3, 6) so that the circle intersects both axes.

Find the center and the radius of the circle given by the equation 
\[(x - 2)^2 + y^2 = 16\]

Find the center and the radius of the circle given by the equation 
\[(x + 1)^2 + (y - 3)^2 = 9\]

Find the center and the radius of the circle given by the equation 
\[0 = x^2 + y^2 - 4x + 2y - 11\]

(Do this, you must complete the squares.)

Find the center and the radius of the circle given by the equation 
\[x^2 + y^2 + 2x - 4y + 1 = 0\]

(Do this, you must complete the squares.)

1.1.4

Convert 75° to radian measure.

Convert \(\frac{5\pi}{12}\) to degree measure.

Convert -15° to radian measure.

Convert \(\frac{3\pi}{4}\) to degree measure.

Evaluate the following expressions without using a calculator:

(a) \(\sin(\frac{-\pi}{3})\)  
(b) \(\cos(\frac{\pi}{6})\)  
(c) \(\tan(\frac{\pi}{4})\)

Evaluate the following expressions without using a calculator:

(a) \(\sin(\frac{\pi}{4})\)  
(b) \(\cos(\frac{\pi}{4})\)  
(c) \(\tan(\frac{\pi}{4})\)

(a) Find the values of \(\alpha \in [0, 2\pi]\) that satisfy

\[\sin \alpha = \frac{-1}{2}\sqrt{3}\]

(b) Find the values of \(\alpha \in [0, 2\pi]\) that satisfy

\[\tan \alpha = \sqrt{3}\]

(a) Find the values of \(\alpha \in [0, 2\pi]\) that satisfy

\[\cos \alpha = \frac{-1}{2}\sqrt{2}\]

(b) Find the values of \(\alpha \in [0, 2\pi]\) that satisfy

\[\sec \alpha = 2\]

Show that the identity

\[1 + \tan^2 \theta = \sec^2 \theta\]

follows from

\[\sin^2 \theta + \cos^2 \theta = 1\]

Show that the identity

\[1 + \cot^2 \theta = \csc^2 \theta\]

follows from

\[\sin^2 \theta + \cos^2 \theta = 1\]

Solve \(2 \cos \theta \sin \theta = \sin \theta\) on \([0, 2\pi]\).

Solve \(\sec^2 x = \sqrt{3} \tan x + 1\) on \([0, \pi]\).

1.1.5

Evaluate the following exponential expressions:

(a) \(4^4\cdot 2^2\)  
(b) \(\frac{4^3 \cdot 2^2}{5^2}\)  
(c) \(\frac{4^{\frac{1}{2}} \cdot 2^3}{5^2}\)

Evaluate the following exponential expressions:

(a) \(2^{\frac{2}{3}} \cdot 2^2\)  
(b) \(\frac{2^{\frac{2}{3}} \cdot 2^2}{5^2}\)  
(c) \(\frac{2^{\frac{3}{2}} \cdot 2^3}{5^2}\)

Which real number \(x\) satisfies

(a) \(\log_{10} x = -2\)  
(b) \(\log_{10} x = 3\)  
(c) \(\log_{10} x = -2\)

Which real number \(x\) satisfies

(a) \(\log_{10} x = -2\)  
(b) \(\log_{10} x = 3\)  
(c) \(\log_{10} x = 3\)

Which real number \(x\) satisfies

(a) \(\log_{10} 32 = x\)  
(b) \(\log_{10} 81 = x\)  
(c) \(\log_{10} 0.001 = x\)

Which real number \(x\) satisfies

(a) \(\log_{10} 64 = x\)  
(b) \(\log_{10} 625 = x\)  
(c) \(\log_{10} 0.001 = x\)

Simplify the following expressions:

(a) \(-\ln 2\)  
(b) \(\log_{10} (\ln x - 4)\)  
(c) \(\log_{10} 4^{x^{-1}}\)

Simplify the following expressions:

(a) \(-\ln 2\)  
(b) \(\ln \frac{2^{x-1}}{2^x}\)  
(c) \(\log_{10} 3^{x+1}\)

Solve for \(x\):

(a) \(e^{3x} = 2\)  
(b) \(e^x = 10\)  
(c) \(e^{x-1} = 10\)

Solve for \(x\):

(a) \(3^x = 81\)  
(b) \(2^{x+1} = 27\)  
(c) \(10^x = 1000\)

Solve for \(x\):

(a) \(\ln (x - 3) = 5\)  
(b) \(\ln (x + 2) + \ln (x - 2) = 1\)

(c) \(\log_{10} x^2 - \log_{10} 3x = 2\)

Solve for \(x\):

(a) \(\ln (2x - 3) = 0\)  
(b) \(\log_{10} (1 - x) = 3\)

(c) \(\ln x - 2 \ln x = 1\)

1.1.6

In Problems 85-92, simplify each expression and write it in the standard form \(a + bi\):

85. \((3 - 2i) - (-2 + 5i)\)  
86. \((7 + i) - 4\)

87. \((4 - 2i) + (9 + 4i)\)  
88. \((6 - 4i) + (2 + 5i)\)

89. \((5 + 3i)\)  
90. \((2 - 3i) + (5 + 2i)\)

91. \((-6 - i)(6 + i)\)  
92. \((-4 - 3i)(4 + 2i)\)

In Problems 93-98, let \(z = 3 - 2i, u = -4 + 3i, v = 3 + 5i,\) and \(w = 1 + i\). Compute the following expressions:

93. \(\overline{z}\)  
94. \(z + u\)  
95. \(\overline{z} + \overline{u}\)

96. \(v - w\)  
97. \(w\)  
98. \(\overline{w}\)

99. If \(z = a + bi,\) find \(z + \overline{z}\) and \(z - \overline{z}\).

100. If \(z = a + bi,\) find \(z\). Use your answer to compute \(\overline{z}\), and compare your answer with \(z\).

In Problems 101-106, solve each quadratic equation in the complex number system:

101. \(2x^2 - 3x + 2 = 0\)  
102. \(3x^2 - 2x + 1 = 0\)

103. \(-x^2 + x + 2 = 0\)  
104. \(-2x^2 + x + 3 = 0\)

105. \(4x^2 - 3x + 1 = 0\)  
106. \(-2x^2 + 4x - 3 = 0\)

In Problems 107-112, first determine whether the solutions of each quadratic equation are real or complex without solving the equation. Then solve the equation:

107. \(3x^2 - 4x - 7 = 0\)  
108. \(3x^2 - 4x + 7 = 0\)

109. \(-x^2 + 2x - 1 = 0\)  
110. \(4x^2 - x + 1 = 0\)

111. \(3x^2 - 5x + 6 = 0\)  
112. \(-x^2 + 7x - 2 = 0\)

113. Show \((\overline{z}) = z\).

114. Show \(\overline{z + w} = \overline{z} + \overline{w}\).

115. Show \(\overline{z} = \overline{z}\).