6.2: The Fundamental Theorem of Calculus

Example: Evaluate each integral by interpreting it in terms of area.

\[(i) \int_{1}^{3} c \, dt \qquad (ii) \int_{a}^{x} c \, dt \qquad (iii) \int_{0}^{x} t \, dt\]

Let \( F(x) = \int_{a}^{x} f(u) \, du \), how does the area change as \( x \) changes?

\[
\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left[ \int_{a}^{x+h} f(u) \, du - \int_{a}^{x} f(u) \, du \right] \\
= \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(u) \, du
\]

What does \( \int_{x}^{x+h} f(u) \, du \) mean graphically? It is about the area under the curve \( f(u) \) from \( x \) to \( x+h \). That is about \( f(x) \cdot h \).

\[
\frac{d}{dx} F(x) = \lim_{h \to 0} \frac{1}{h} \int_{x}^{x+h} f(u) \, du = \lim_{h \to 0} \frac{1}{h} \cdot f(x) \cdot h = \lim_{h \to 0} f(x) = f(x)
\]
THE FUNDAMENTAL THEOREM OF CALCULUS, PART 1

If $f$ is continuous on $[a,b]$, then the function $F$ defined by

$$F(x) = \int_a^x f(u) \, du, \quad a \leq x \leq b$$

is continuous on $[a,b]$, differentiable on $(a,b)$ and $F'(x) = f(x)$. So $F(x)$ is an antiderivative of $f(x)$.

Example: Find the derivative of the given functions

(i) $g(x) = \int_{-1}^x \sqrt{t^3 + 1} \, dt$  
(ii) $y = \int_{-1}^x \frac{2}{t^2 + 1} \, dt$

THE FUNDAMENTAL THEOREM OF CALCULUS, PART 2

If $f$ is continuous on $[a,b]$, then

$$\int_a^b f(x) \, dx = F(b) - F(a) = F(x)|_a^b$$

where $F$ is any antiderivative of $f$. 

**Example:** Evaluate the following definite integrals

(i) \[ \int_0^2 (w^3 - 1)^2 \, dw \]
(ii) \[ \int_{-2}^0 |x^2 - 1| \, dx \]
(iii) \[ \int_0^{\pi/2} (\cos \theta + 2 \sin \theta) \, d\theta \]
If the general antiderivative of \( f(x) \) is \( F(x) + C \), this can be written as
\[
F(x) + C = C + \int_{a}^{x} f(u) \, du = \int f(x) \, dx
\]

Where \( \int f(x) \, dx \) is called an **indefinite integral**.

**Example**: Compute the following indefinite integrals

(i) \( \int (x^{3/5} + x^{5/3}) \, dx \)  
(ii) \( \int \cos \left( \frac{2 - 4x}{5} \right) \, dx \)  
(iii) \( \int 2e^{-x/4} \, dx \)

(iv) \( \int 4^{-x} \, dx \)  
(v) \( \int \frac{5}{\sqrt{1-x^2}} \, dx \)