5.2: Monotonicity and Concavity

A function $f$ defined on an interval $I$ is called (strictly) increasing on $I$ if
\[ f(x_1) < f(x_2) \] whenever $x_1 < x_2$ in $I$

and is called (strictly) decreasing on $I$ if
\[ f(x_1) < f(x_2) \] whenever $x_1 < x_2$ in $I$

A function that is always increasing or always decreasing is called monotonic.

First Derivative Test for Monotonicity

Suppose $f$ is continuous on $[a,b]$ and differentiable on $(a,b)$
- If $f'(x) > 0$ for all $x \in (a,b)$, then $f$ is increasing on $[a,b]$
- If $f'(x) < 0$ for all $x \in (a,b)$, then $f$ is decreasing on $[a,b]$

Second Derivative Test for Concavity

Suppose $f$ is twice differentiable on an open interval $I$
- If $f''(x) > 0$ for all $x \in I$, then $f$ is concave up on $I$
- If $f''(x) < 0$ for all $x \in I$, then $f$ is concave down on $I$

A critical number of a function $f$ is a number $c$ in the domain of $f$ such that either $f'(c) = 0$ or $f'(c)$ does not exist.

An inflection point of a function $f$ is the point where a function changes concavity.
Where is the function increasing? decreasing?
Where does the function have a local maximum? local minimum?
Where is the function concave up? concave down?
Where are the critical numbers and inflection points?
Example: The graph of the derivative of $f$ is shown.

(a) Where is the function increasing or decreasing?

(b) Where might the function have a local maximum or minimum?

(c) Where is the function concave up or concave down?

(d) Where are the inflection points?

(e) If $f(0) = 0$, sketch a possible graph of $f$. 
Example: Sketch a graph of $f$ satisfying the following conditions:

$$f'(x) > 0 \text{ on } (-\infty, 1) \text{ and } f'(x) < 0 \text{ on } (1, \infty)$$
$$f''(x) > 0 \text{ on } (-\infty, -2) \text{ and } (2, \infty)$$
$$f''(x) < 0 \text{ on } (-2, 2)$$
$$\lim_{x \to -\infty} f(x) = -2 \text{ and } \lim_{x \to \infty} f(x) = 0$$

Example: Determine where each function is increasing, decreasing, concave up, and concave down.

(a) $y = (3x - 1)^{1/3}$

(b) $y = \frac{-2}{x^2 + 3}$