Chapter 5: Applications of Differentiation

5.1: Extrema and the Mean-Value Theorem

Absolute Extreme values of $f$
- A function $f$ has an absolute (or global) maximum at $c$ if $f(c) \geq f(x)$ for all $x$ in the domain of $D$. The number $f(c)$ is called the maximum value of $f$ on $D$.
- A function $f$ has an absolute (or global) minimum at $d$ if $f(d) \leq f(x)$ for all $x$ in the domain of $D$. The number $f(d)$ is called the minimum value of $f$ on $D$.

Local Extreme values of $f$
- A function $f$ has a local (or relative) maximum at $c$ if $f(c) \geq f(x)$ when $x$ is near $c$.
- A function $f$ has a local (or relative) minimum at $d$ if $f(d) \leq f(x)$ when $x$ is near $c$.

The Extreme Value Theorem
If $f$ is continuous on a closed interval $[a,b]$, then $f$ attains an absolute maximum value of $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a,b]$.
Example: The graph of the function $f$ with domain $[-5,5]$ is shown.

(a) What are the absolute extrema?

(b) What are the local extrema?

Example: Find all absolute and local extrema for the following functions

$$f(x) = 1 - x^2, \ 0 \leq x \leq 1$$

$$f(x) = 1 - x^2, \ -1 \leq x \leq \frac{1}{2}$$
\[ f(x) = \frac{1}{x}, \quad 0 < x < 1 \]

\[ f(x) = \begin{cases} 
  x^2 & \text{for } -1 \leq x < 0 \\
  2 - x^2 & \text{for } 0 \leq x \leq 1 
\end{cases} \]

**Fermat’s Theorem**

If \( f \) has a local maximum or minimum at \( c \), and if \( f'(c) \) exists, then \( f'(c) = 0 \).

**Critical Numbers**

A critical number (or value) of a function \( f \) is a number \( c \) in the domain of \( f \) such that either \( f'(c) = 0 \) or \( f'(c) \) does not exist.
Example: Find all critical numbers for the following functions

\[ f(x) = x^3 + 6x^2 + 3x - 1 \]

\[ f(x) = |x^2 - 1| \]

\[ f(x) = \frac{x + 1}{x^2 + x + 1} \]

\[ f(x) = x + \sin x \]

\[ f(x) = xe^{2x} \]

\[ f(x) = \sqrt[3]{x^2 - x} \]
Finding Absolute Extrema

To find the absolute extrema of a continuous function $f$ on a closed interval $[a,b]$,

1. Find all critical values $c_1, c_2$, etc of $f$ in the interval $(a,b)$
2. Find the values of $f(c_1), f(c_2)$ etc along with $f(a)$ and $f(b)$
3. The absolute maxima is the largest of the values found in steps 1 and 2.
4. The absolute minima is the smallest of the values found in steps 1 and 2.

Example: Find the absolute extrema for the following functions

i. $f(x) = x^3 - 12x + 1$, $[-3, 5]$

ii. $f(x) = -x^3 + 27x + 1$, $[0, 4]$

iii. $f(x) = x - 2\cos x$, $[0, \pi]$
iv. \( f(x) = x - 2\cos x, \quad [-\pi, \pi] \)

v. \( f(x) = \frac{x}{x+1}, \quad [1,2] \)

vi. \( f(x) = \frac{x}{x+1}, \quad [-2,2] \)

vii. \( f(x) = \frac{\ln x}{x}, \quad [1,3] \)
Mean Value Theorem
If \( f \) is continuous on a closed interval \([a,b]\) and differentiable on the interval \((a,b)\), then there exists a number \( c \), where \( a < c < b \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

Example: Given \( f(x) = 4 - x^2 \), show \( f(x) \) satisfies the MVT on \([1,2]\) and find all values of \( c \) that satisfy the conclusion of the MVT.

Example: Given that \( 1 \leq f'(x) \leq 4 \) for all \( x \) in the interval \([2,5]\), prove that \( 3 \leq f(5) - f(2) \leq 12 \) given that \( f \) is continuous on \([2,5]\) and differentiable on \((2,5)\)