Chapter 4: Differentiation

4.1 Formal Definition of the Derivative

What is the slope of the secant line $PQ$?

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through $P$ with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

The equation of the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is given by

$$y - f(a) = m(x - a)$$

The normal line is defined as the line that is perpendicular to the tangent line at the point of tangency. The slope of the normal line to the graph of $f(x)$ is $-\frac{1}{m}$.
Example:
a) Find the slope of the tangent line to the curve $y = x^3$ at $(-1, -1)$ using both definitions given above.
b) Find the equation of the tangent line.
c) Graph the curve and the tangent line
Example: Find the equation of the tangent line and the normal line to the curves below at the given point.

a) \( y = \frac{x}{1-x} \), \((0,0)\)  
b) \( y = \frac{1}{\sqrt{x}} \), \((1,1)\)
The **average rate of change** of \( y = f(x) \) with respect to \( x \) is
\[
\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]
The **instantaneous rate of change** of \( y = f(x) \) with respect to \( x \) at the point \( x = x_1 \) is
\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}
\]

**Example:** The population \( P \) (in thousands) of a city from 1990 to 1996 is given in the following table:

<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>( P )</td>
<td>105</td>
<td>110</td>
<td>117</td>
<td>126</td>
<td>137</td>
<td>150</td>
<td>164</td>
</tr>
</tbody>
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(a) Find the average rate of growth from

(i) 1992 to 1996

(iv) 1992 to 1993

(b) Estimate the instantaneous rate of growth in 1992 by measuring the slope of a tangent.
If the position of an object at time $t$ is given by the function $s = f(t)$, then the velocity of the function at time $t = a$ is

$$v(a) = \lim_{{h \to 0}} \frac{f(a + h) - f(a)}{h}$$

**Example:** The displacement (in meters) of a particle moving in a straight line is given by $s = t^2 - 8t + 18$ where $t$ is measured in seconds.

a) Find the average velocity over the following time intervals

- $[3,4]$
- $[3.5,4]$
- $[4,5]$
- $[4,4.5]$

b) Find the instantaneous velocity when $t = 4$

c) Draw the graph of $s$ as a function of $t$ and draw the secant and tangent lines from parts (a) and (b).
Example: If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in 1 hour, then Torricelli’s Law gives the volume $V$ of water remaining in the tank after $t$ minutes as

$$V(t) = 100,000 \left(1 - \frac{t}{60}\right)^2 \quad 0 \leq t \leq 60$$

Find the rate at which the water is flowing out of the tank after 20 minutes.
The **derivative** of a function \( f(x) \) at a number \( a \), denoted by \( f'(a) \)

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

**Interpretations**
- Slope of the tangent line at \( x = a \)
- Instantaneous rate of change at \( x = a \)
- Velocity at \( x = a \)

**Notation:** \( f'(x) = \frac{d}{dx} f(x) = \frac{df}{dx} = \frac{dy}{dx} = y' \)

**Example:** Find \( f'(a) \) for the given functions
a) \( f(x) = x^3 + 5x + 2 \) at \( a = 1 \)       b) \( f(x) = \sqrt{x - 1} \) at \( a = 5 \)
Example: Each limit represents the derivative of some function $f$ at some number $a$. State $f$ and $a$ in each case.

a) $\lim_{h \to 0} \frac{(2+h)^3 - 8}{h}$

b) $\lim_{x \to 3\pi} \frac{\cos x + 1}{x - 3\pi}$

Given a function $f(x)$, we associate with it a new function $f'$, called the derivative of $f$ defined by the equation below:

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

A function $f$ is **differentiable** at $a$ if $f'(a)$ exists. It is differentiable on an open interval $(a, b)$ if it is differentiable at every number in the interval. If $f$ is differentiable at $a$, then $f$ is continuous at $a$.

Example: At what values of $x$ is $f$ not differentiable?
**Example:** Match the graph of each function in (a) – (d) with the graph of its derivation in I – IV. Explain

(a)  

(b)  

(c)  

(d)  

(I)  

(II)  

(III)  

(IV)  

**Example:** Find the derivative of the given functions using the definition of derivative. State the domain of the function and the domain of its derivative.

(a) \( f(x) = \sqrt{6 - x} \)  
(b) \( f(x) = \frac{x + 1}{x - 1} \)  
(c) \( g(x) = \frac{1}{x^2} \)  
(d) \( G(t) = \frac{1}{\sqrt{t - 1}} \)