3.5 Properties of Continuous Functions

**Intermediate Value Theorem**
Suppose \( f \) is continuous on the closed interval \([a, b]\) and let \( N \) be any number strictly between \( f(a) \) and \( f(b) \). Then there exists a number \( c \) in \((a, b)\) such that \( f(c) = N \).

**Example:** Use the Intermediate Value Theorem to show there is a root of the given equation in the given interval.

a) \( x^3 - 2x^2 - x - 3 = 0 \), \((2, 3)\)
\[ f(x) = x^3 - 2x^2 - x - 3 \] is a polynomial and therefore continuous on \( \mathbb{R} \).
\[ f(2) = -5 \] and \( f(3) = 75 \)
Since there is a sign change on \([2, 3]\), there is a root in \((2, 3)\) where \( f(x) = 0 \).

b) \( x^2 = \sqrt{x+1} \), \((1, 2)\)
\[ f(x) = x^2 - \sqrt{x+1} \] is continuous for \( x > -1 \) so I VT say there is a root to \( x^2 - \sqrt{x+1} = 0 \) if there is a sign change.
\[ f(1) = 1^2 - \sqrt{1+1} = 1 - \sqrt{2} < 0 \] and \( f(2) = 2^2 - \sqrt{2+1} = 4 - \sqrt{3} > 0 \) so \( f(x) = x^2 - \sqrt{x+1} = 0 \) has a root on \((1, 2)\).

**Example:** Use the Intermediate Value Theorem to show that there is a positive number \( c \) such that \( c^2 = 2 \).

In which of the intervals below does \( y = -x^3 + 4x^2 - 5x + 3 \) have a root?
(A) \([-1, 0]\)  (B) \([0, 1]\)  (C) \([1, 2]\)  (D) \([2, 3]\)  (E) None of these contain a root.

\[ y(-1) = -(-1)^3 + 4(-1)^2 - 5(-1) + 3 = 13 \]
\[ y(0) = 3 \]
\[ y(1) = 1 \]
\[ y(2) = 1 \]
\[ y(3) = -3 \]