3.2 Continuity

**Example:**
Where is the graph discontinuous?

\[ f(x) \text{ is continuous at a number } a \text{ if } \lim_{x \to a} f(x) = f(a) \]

\[ f(x) \text{ is continuous from the right at a number } a \text{ if } \lim_{x \to a^+} f(x) = f(a) \]

\[ f(x) \text{ is continuous from the left at a number } a \text{ if } \lim_{x \to a^-} f(x) = f(a) \]

So to check for continuity, do the following:
1. Make sure \( f(x) \) is defined at \( x = a \)
2. Check that \( \lim_{x \to a} f(x) \) exists
3. Finally, check that \( \lim_{x \to a} f(x) = f(a) \)
Example: Explain why each function is discontinuous at the given point.

a) \( a = -1 \), \( f(x) = \frac{x^2 - 1}{x + 1} \)

b) \( a = 4 \), \( f(x) = \begin{cases} \frac{x^2 - 2x - 8}{x - 4} & \text{if } x \neq 4 \\ 3 & \text{if } x = 4 \end{cases} \)

A function \( f(x) \) is continuous on an interval if it is continuous at every number in the interval. At the endpoints it is understood that the function is left or right continuous.

**Continuity Rules**

If \( f \) and \( g \) are continuous at \( a \) and \( c \) is a constant, then the following functions are also continuous at \( a \)

\[ f \pm g \quad cf \quad fg \quad \frac{f}{g} \quad (\text{provided that } g(a) \neq 0) \]

The following are continuous wherever they are defined:

1. Polynomials
2. Rational functions
3. Power functions
4. Trigonometric functions
5. Exponential functions of the form \( a^x \), \( a > 0 \), and \( a \neq 1 \)
6. Logarithmic function of the form \( \log_a x \), \( a > 0 \), and \( a \neq 1 \)
Example: Find where the given functions are discontinuous.

(a) \[ g(x) = \begin{cases} 
2x + 1 & \text{if } x \leq -1 \\
3x & \text{if } -1 < x < 1 \\
2x + 1 & \text{if } x \geq 1 
\end{cases} \]

(b) \[ h(t) = \frac{t + 1}{4t^2 - 1} \]

c) \[ h(x) = \begin{cases} 
\sqrt{-x} & \text{if } x \leq 0 \\
1 & \text{if } 0 < x < 1 \\
\sqrt{x} & \text{if } x \geq 1 
\end{cases} \]
Example: Find the value or values of $c$ that make $f$ continuous

$$f(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4 \\ cx + 20 & \text{if } x \geq 4 \end{cases}$$

If $g(x)$ is continuous at $x = a$ with $g(a) = L$, and $f(x)$ is continuous at $x = L$, then $(f \circ g)(x)$ is continuous at $x = a$. That is,

$$\lim_{x \to a} (f \circ g)(x) = \lim_{x \to a} f[g(x)] = f\left[\lim_{x \to a} g(x)\right] = f[g(a)] = f(L)$$

Example: Find where the given functions are discontinuous

a) $F(x) = 2x + \sqrt{25 - x^2}$

b) $g(t) = \frac{1}{t + \sqrt{t^2 - 4}}$

c) $G(x) = |x - x^2|$

d) $H(x) = \sqrt{\frac{x - 2}{5 + x}}$
A removable discontinuity at $a$ is a discontinuity that can be removed by redefining the function at $a$.

**Example:**
Which of the following functions $f$ has a removable discontinuity at $a$? If it is removable, redefine $f$ so that the function is continuous on $\mathbb{R}$.

a) $a = -2, \quad f(x) = \frac{x^2 - 2x - 8}{x + 2}$

b) $a = 7, \quad f(x) = \frac{x - 7}{|x - 7|}$
c) \( a = -4, \quad f(x) = \frac{x^3 + 64}{x + 4} \)

d) \( a = 9, \quad f(x) = \frac{3 - \sqrt{x}}{9 - x} \)