Ch 2: Discrete-Time Models, Sequences, and Difference Equations

2.1: Exponential Growth and Decay

2.1.1: Modeling Population Growth in Discrete Time

Example: Consider a colony of 6 bacteria that will double every 40 minutes. Find an equation that shows the number of bacteria with the time unit being $t = 1$ hour.

\[
\begin{align*}
\text{time (min)} &\quad 0 &\quad 40 &\quad 80 &\quad 120 \\
\text{unit (hr)} &\quad 0 &\quad 0.67 &\quad 1.33 &\quad 2.00 \\
\end{align*}
\]

\[
\begin{aligned}
N_0 &= 6 \\
R &= 2^{\frac{1}{2}}
\end{aligned}
\]

A function of the form $N_0R^t$ is an exponential function. Since our focus is on biological models, $R > 0$ and $N_0 \geq 0$. $R$ is the growth constant. In addition, for $0 < R < 1$ this will model exponential decay, for $R = 1$ this will model no growth or decay, and for $R > 1$ this will model exponential growth.

2.1.2: Recursions

Another way to show that a population triples every time step is to write

\[N_{t+1} = 3N_t \quad \text{given an } N_0\]

A rule that is repeated to go from one time step to the next is called a recursion and we say that the population is defined recursively.
Example: Consider a population $N_0$ that triples every time step. Show this relationship recursively and as an exponential function. What is the growth constant? Show this relationship graphically by plotting $N_t$ on the horizontal axis and $N_{t+1}$ on the vertical axis with $N_0 = 1$.

$$N_0 = 1, \quad N_1 = 3 \cdot 1 = 3, \quad N_2 = 3 \cdot 3 = 3^2, \quad N_3 = 3 \cdot 3^2 = 3^3 \Rightarrow N(t) = N_0 (3)^t$$

$$N(t) = N_0 3^t \quad \Rightarrow \quad N_{t+1} = 3N_t \quad \text{with } N_0 \text{ starting recursive relationship}$$

$R = 3 = \text{growth constant}$

$(N_0, N_1) = (1, 3)$

$(3, 9), (9, 27)$

Time is the implicit variable

Graph $N_{t+1} = \frac{1}{4} N_t$ with $N_0 = 16$

$$N_0 = 16, \quad N_1 = 16 \cdot \frac{1}{4} = 4, \quad N_2 = 1, \quad N_3 = \frac{1}{4}$$

$(16, 4), (4, 1), (1, \frac{1}{4})$