1.2 Elementary Functions

Let $D$ and $R$ be two nonempty sets. A function $f$ from $D$ to $R$ is a rule that assigns to each element $x$ in $D$ one and only one element $y = f(x)$ in $R$.

The set $D$ in the definition is called the **domain** of $f$ and it is the set of possible inputs for our function. The letter representing the elements in the domain is called the **independent variable**. In the definition above, the independent variable is $x$.

The set $R$ is called the range of $f$ and it is the set of all possible values of the output of our function. The letter representing the elements in the domain is called the **dependent variable**. In the definition above, the independent variable is $y$.

The **graph** of a function $f$ consists of all points $(x, y)$ such that $x$ is in the domain of $f$ and $y = f(x)$. A graph in the $xy$-plane represents a function if and only if every vertical line intersects the graph in at most one place. This is called the **vertical line test**.

**Example:** Which of the graphs below are graphs of functions?
A **polynomial** is a function of the form

\[ f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \]

where \( n \) is a non-negative integer and \( a_i \) are real valued constants. The **leading coefficient** is \( a_n \) and \( a_n \neq 0 \). The **degree** of the polynomial is \( n \).

The domain of a polynomial is \( \mathbb{R} \) or \(( -\infty, \infty )\).

Some common polynomials

**degree 0:** \( y = 2.5 \)
- constant function

**degree 1:** \( y = 2x - 3 \)
- linear function

**degree 2:** \( y = x^3 - 2x^2 \)
- quadratic function

**degree 3:** \( y = -x^2 + 3x + 4 \)
- cubic function

A rational function \( f(x) \) is the quotient of two polynomial functions \( p(x) \) and \( q(x) \):

\[ f(x) = \frac{p(x)}{q(x)} \]

The domain of \( f(x) \) is all real numbers for which \( q(x) \neq 0 \)

**Example:** What is the domain of \( f(x) = \frac{x + 1}{x^2 - 1} \)?
A function of the form \( f(x) = x^r \) where \( r \) is a real number is called a **power function**. Some examples of power functions:

\[
\begin{align*}
y &= x^{-1} \\
y &= x^{\frac{1}{2}}
\end{align*}
\]

**Example:** In the case of \( f(x) = \sqrt{x} \) what is the domain?

\( f(x) = a^x, \ a > 0, \ a \neq 1 \) is called an **exponential function**. Exponential functions are often used to model growth and decay. If \( A_0 \) is the initial amount and \( k \) is the growth or decay rate, then the population at time \( t \) is given by

\[
A(t) = A_0 e^{kt}
\]

For exponential growth, \( k > 0 \) and for exponential decay, \( k < 0 \).

**Example:** A bacteria culture starts with 4000 bacteria and the population triples every 30 minutes. Find an expression for the number of bacteria after \( t \) hours.
The **half-life** of a substance is the amount of time it takes for half of the substance to disintegrate.

*Example:* After 3 days a sample of an unknown radioactive element is found to have decayed to 58% of its original amount. What is the half-life of this element?  [http://ie.lbl.gov/education/isotopes.htm](http://ie.lbl.gov/education/isotopes.htm)

A function $f(x)$ is **one-to-one** provided that whenever $f(x_1) = f(x_2)$ then $x_1 = x_2$. One way to check if a function is one-to-one is to use the horizontal line test.

Let $f(x)$ be a one-to-one function with domain $D$ and range $R$. Then the **inverse** function $f^{-1}(x)$ exists. The domain of $f^{-1}(x)$ is $R$ and the range of $f^{-1}(x)$ is $D$. Moreover,

$$f(x) = y \iff f^{-1}(y) = x$$

*Example:* Find the inverse, and the domain and range of $f(x) = 5 - 4x^3$.  
The inverse of the exponential function \( f(x) = a^x \) is \( f^{-1}(x) = \log_a x \).

Example: Find the domains and ranges of \( f(x) = a^x \) and \( f^{-1}(x) = \log_a x \)

\[
y = e^x \quad \text{and} \quad y = \ln x
\]

\[
y = \left(\frac{1}{2}\right)^x \quad \text{and} \quad y = \log_{\frac{1}{2}} x
\]

*Example:* Find the domain and range of each function

\[ f(x) = \log\left(x^2 - 3x + 2\right) \]
\[ g(x) = \ln\left(x^3 - x\right) \]

*Example:* Write the expressions \( 5^x \) and \( \log_7(x + 5) \) in terms of base \( e \).
$f(x)$ is a **periodic** function if there is a positive constant $c$ such that $f(x+p) = f(x)$. If $p$ is the smallest number with this property, we call it the **period** of $f(x)$. The trigonometric functions are periodic functions.

For any real number $a$ and $k \neq 0$, the functions $f(x) = a \sin(kx)$ and $g(x) = a \cos(kx)$ have an amplitude of $|a|$ and a period of $p = \frac{2\pi}{|k|}$.

**Example:** Find the amplitude and period of $f(x) = 3 \sin(2x)$ and graph the function.