Clark measures with prescribed behavior and rank-one perturbations of self-adjoint and unitary operators

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Let \( \varphi \) be an analytic map of the unit disc \( \mathbb{D} \subset \mathbb{C} \) to itself for which \( \varphi(0) = 0 \). For each \( \alpha \in \partial \mathbb{D} \) the function
\[
\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)}
\]
has positive real part and hence a Herglotz integral representation
\[
\frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \int \frac{\zeta + z}{\zeta - z} d\mu_\alpha(\zeta)
\]
for a uniquely determined probability measure \( \mu_\alpha \) on \( \partial \mathbb{D} \). The measures \( \{\mu_\alpha\}_{\alpha \in \partial \mathbb{D}} \) are called the Clark measures associated to \( \varphi \) after the paper [1] by D. N. Clark, though other names are common in the literature.

If we define the unitary operators \( \{U_\alpha\}_{\alpha \in \partial \mathbb{D}} \) to be multiplication by the coordinate function \( z \) on \( L^2(\mu_\alpha) \) then a calculation shows
\[
U_\alpha f(z) = U f(z) + (\alpha - 1) \int f d\mu
\]
where \( U = U_1 \). Thus \( \{U_\alpha\}_{\alpha \in \partial \mathbb{D}} \) is a family of unitary rank-one perturbations of the unitary operator \( U \); with spectral measures \( \mu_\alpha \). Such families and analogous families of rank-one perturbations of self-adjoint operators have been much-studied in the mathematical physics community, see e.g [6]. A question of interest is the extent to which properties of the spectral measures \( \mu_\alpha \) can vary with \( \alpha \). A standard calculation shows that the absolutely continuous parts of the \( \mu_\alpha \)'s are all mutually absolutely continuous, so we focus on the singular parts \( \sigma_\alpha \). In this context we note that if \( \varphi \) is inner then all \( \mu_\alpha \)'s are singular.

It came as a bit of a surprise that properties of the singular measures \( \sigma_\alpha \) can be very sensitive to changes in \( \alpha \). The first result along this line was due to W. Donoghue [3]. Translated into our language his result says:

**Theorem 1.** There exists an inner function \( \varphi \) whose Clark measures \( \{\mu_\alpha\} \) satisfy

\[\mu = \mu_1 \text{ is purely atomic}\]

but

\[\mu_\alpha \text{ is continuous singular for } \alpha \in \partial \mathbb{D} \setminus \{1\}\]

The measure \( \mu \) in Donoghue's example has all of \( \partial \mathbb{D} \) for its support. The following result of R. Del Rio, N. Makarov, and B. Simon and, independently, A. Ya. Gordon, shows that the "largeness" of the set of \( \alpha \)'s for which \( \mu_\alpha \) is singular continuous is to be expected (we translate their result into our language):

**Theorem 2** ([2], [4], [5]). Let \( \varphi \), \( \{\mu_\alpha\} \) be as above and let \( I \subset \partial \mathbb{D} \) be a closed interval, not a singleton, such that \( I \subset \text{sp} \mu \) and \( \mu|_I \) is singular. Then for all \( \alpha \) in an dense \( G_\delta \)-subset of \( \partial \mathbb{D} \) \( \mu_\alpha|_I \) is singular continuous.

We prove the following converse:
Theorem 3. Let $I$ be a closed subinterval of $\partial \mathbb{D}$, not a singleton, and let $G$ be a $G_\delta$-subset of $\partial \mathbb{D}$. Then there exists an inner function $\varphi$ whose associated Clark measures $\mu_\alpha$ satisfy:

$$spt \mu \subset I$$

$$\mu_\alpha|I \text{ is singular if } \alpha \in G$$

$$\mu_\alpha \text{ is purely atomic if } \alpha \in \partial \mathbb{D} \setminus G$$

If $G$ is dense, then of necessity $spt \mu = 1$.

The function $\varphi$ is produced by constructing a Riemann surface $\mathcal{R}$ lying over $\mathbb{D}$ with projection $\pi : \mathcal{R} \rightarrow \mathbb{D}$, then setting $\varphi = \pi \circ \Phi$, where $\Phi : \mathbb{D} \rightarrow \mathcal{R}$ is a covering map.

References


