On my honor, as an Aggie, I have neither given nor received unauthorized aid on this work.

Name (print):  

In all questions, no analytical work — no points.
1. A styrofoam board is cut in the shape of the right half of the cardioid \( r = 1 - \sin \theta \). A static electricity charge is put on the board whose surface charge density is given by \( \rho_e = x \). Find the total charge \( Q \) on the board, \( Q = \iint \rho_e \, dA \).

\[
Q = \iint x \, dA \\
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\frac{1 - \sin \theta}{2}}^{\frac{1 + \sin \theta}{2}} r \cos \theta \, r \, dr \, d\theta \\
= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u \cos \theta \left( \frac{1 - \sin \theta}{3} \right)^3 \, d\theta \\
= -\int_{0}^{1} \frac{u^3}{3} \, du = -\frac{u^4}{12} = -\frac{(1 - \sin \theta)^4}{12} \\
= 0 + \frac{2^4}{12} = \frac{4}{3}
\]
2. Compute \( \iiint_{0}^{4-x^2-y^2} z \cos \left( (x^2 + y^2 + z^2)^{2} \right) \, dz \, dy \, dx \).

\[
2 = \sqrt{4-x^2-y^2} \\
x^2 + y^2 + z^2 = R \\
Z = \rho \cos \phi
\]

\[
\frac{\pi}{2} \frac{1}{2} z \int_{0}^{\pi/2} \int_{0}^{\pi} \rho \sin \phi \cos(\rho^4) \rho^2 \sin \phi \, d\phi \, d\theta \\
= \frac{\pi}{2} \int_{0}^{\pi/2} \sin 2\phi \, d\phi \left( \frac{1}{4} \int_{0}^{\pi} \cos(\rho^4) \, d\rho^4 \right) \\
= \frac{\pi}{2} \left[ \sin 2\phi \right]_{0}^{\pi/2} \left[ \frac{1}{4} \rho^4 \right]_{0}^{2} \\
= \frac{\pi}{2} \frac{2}{4} \frac{\sin 16}{4} = \frac{\pi \sin 16}{16}
\]
3. Compute \( \iint_R \frac{y}{x} dA \) over the diamond-shaped region \( R \) bounded by the curves \( y = x, \ y = x/3, \ y = 1/x, \ y = 2/x \).

\[
\frac{y}{x} = 1 \quad \frac{y}{x} = \frac{1}{3} \quad yx = 1 \quad yx = 2
\]

\[
\frac{y}{x} = u \quad yx = v
\]

\[
y = \sqrt{uv} = u^{1/2} v^{1/2} \quad x = \frac{v}{u}
\]

\[
x = \sqrt{u} u^{-1/2}
\]

\[
J = \begin{vmatrix}
-\frac{1}{2} u^{-3/2} v^{1/2} & \frac{1}{2} u^{-1/2} v^{1/2} \\
\frac{1}{2} u^{-1/2} v^{1/2} & \frac{1}{2} u^{1/2} v^{-1/2}
\end{vmatrix}
= -\frac{1}{4} u^{-1} - \frac{1}{4} u^{-1}
\]

\[
= -\frac{1}{2u}
\]

\[
\iint_R \frac{y}{x} dA = \int_1^2 \int_{1/3}^1 u \cdot \frac{1}{2u} \ du \ dv
\]

\[
= 1 \cdot \frac{2 \cdot \frac{1}{2}}{3} = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}
\]

Also possible by polar

\[
tan^2(\frac{\pi}{6}) \leq \theta \leq \frac{\pi}{4}
\]

\[
1 \leq r \leq \frac{\sqrt{2}}{\sqrt{\sin \theta \cos \theta}}
\]

or change

\[
u = \frac{u}{x} \quad v = x
\]

or

\[
u^2 = \frac{y}{x} \quad v^2 = yx
\]
4. Evaluate the line integral \( \int_C z^2 \, dx - z \, dy + 2y \, dz \) over the line segment from \((0, 0, 0)\) to \((1, -1, 2)\).

\[
\begin{align*}
\chi &= t & y &= -t & z &= 2t \\
\frac{dx}{dt} &= 1 & \frac{dy}{dt} &= -1 & \frac{dz}{dt} &= 2 \\
\int_0^1 4t^2 \, dt + 2t \, dt &= 2t \cdot 2 \, dt \\
&= \frac{4}{3} + 1 - 2 = \frac{1}{3}
\end{align*}
\]
5. Bonus question +2 points: Find the center of mass of the semi-circular wire \( x^2 + y^2 = 1 \), \( y \geq 0 \) (use symmetry!). You want to pass the wire above a fence of height \( h \). You cannot bend the wire. You can rotate and move it in space as you please, as long as the center of mass of the wire must remain at or below the height \( z \). What's the highest fence you can achieve this with? Illustrate with diagrams!

\[
M = \int A \, ds \quad \quad \quad \quad x = \cos t \\
= \int_0^\pi \sqrt{(-\sin^2 t)^2 + (\cos t)^2} \, dt = \frac{\pi}{2}
\]

\[
M \cdot \bar{y} = \int y \, ds = \int_0^\pi \sin t \, dt = -\cos t \bigg|_0^\pi = 2
\]

\[
g = \frac{2}{\pi}
\]

\[
h = 2 + g = 3 - \frac{2}{\pi^2}
\]

Points: \( /20 \)