Homework 7: Sections 3.10, 3.11, & 4.1

Show ALL work on your own paper to get full credit. This assignment is due at the beginning of class on Wednesday, March 28.

1. (3.10) A hot air balloon takes off vertically and maintains a constant speed of 10 mph. A man is watching the take-off from a point 2 miles away.

(a) At what rate is the angle of elevation from the man to the balloon changing when that angle is \( \pi/6 \)?

\[
\text{Want: } \frac{d\theta}{dt} \text{ when } \theta = \pi/6
\]

\[
\text{Know: } \frac{dy}{dt} = 10 \text{ mph}
\]

\[
\tan \theta = \frac{y}{2}
\]

\[
\sec^2 \theta \cdot \frac{d\theta}{dt} = \frac{1}{2} \frac{dy}{dt}
\]

When \( \theta = \pi/6 \):

\[
\frac{1}{\cos^2 \pi/6} \cdot \frac{d\theta}{dt} = 5
\]

\[
\frac{d\theta}{dt} = 5 \cos^2 \frac{\pi}{6} = 5 \left( \frac{\sqrt{3}}{2} \right)^2 = 5 \left( \frac{3}{4} \right) = \frac{15}{4}
\]

\[
\frac{d\theta}{dt} = \frac{15}{4} \text{ rad/hr}
\]

(b) At what rate is the distance from the man to the balloon changing after 15 minutes?

\[
\text{Want: } \frac{dz}{dt} \text{ after 15 mins } = \frac{1}{4} \text{ hr.}
\]

\[
\text{Know: } \frac{dy}{dt} = 10 \text{ mph}
\]

\[
2^2 + y^2 = z^2
\]

\[
2y \cdot \frac{dy}{dt} = 2z \cdot \frac{dz}{dt} \Rightarrow y \cdot \frac{dy}{dt} = z \cdot \frac{dz}{dt}
\]

After 1/4 hr, \( y = 10 \left( \frac{1}{4} \right) = \frac{5}{2} \text{ miles} \)

What is \( z \):

\[
\frac{z^2}{2} = 2^2 + \left( \frac{5}{2} \right)^2 = 4 + \frac{25}{4} = \frac{41}{4}
\]

\[
z = \sqrt{\frac{41}{4}} = \frac{\sqrt{41}}{2} \text{ miles}
\]

So,

\[
\frac{d^2}{dt} = \frac{\sqrt{41}}{2} \text{ mph}
\]
2. (3.10) Water is being poured into an inverted conical tank at a constant rate of 10 m³/min. Water is also leaking out of the tank at some unknown constant rate. The tank has a height of 40 m and a radius of 5 m. If it is known that the water level is decreasing at a rate of 2 m/min when the height of the water is 15 m, find the rate at which the water is leaking out of the tank.

\[ \text{WANT: } \frac{dR}{dt} \quad \text{[Rate out]} \]

\[ \text{Know: } \frac{dV}{dt} = \text{Rate in} - \text{Rate Out} = 10 - \frac{dR}{dt} \text{ m}^3/\text{min} \]

\[ \frac{dh}{dt} = -2 \text{ m/min when } h = 15 \text{ m} \]

\[ V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left( \frac{1}{8} h \right)^2 h \]

\[ V = \frac{1}{192} \pi h^3 \]

\[ \frac{dV}{dt} = \frac{1}{64} \pi h^2 \cdot \frac{dh}{dt} \]

Use given:

\[ \frac{dV}{dt} = \frac{1}{64} \pi (15)^2 (-2) = \frac{-225 \pi}{32} \text{ m}^3/\text{min} \]

Since \( \frac{dV}{dt} = 10 - \frac{dR}{dt} \), then \( \frac{dR}{dt} = 10 - \frac{dV}{dt} \)

So,

\[ \frac{dR}{dt} = 10 - \left( \frac{-225 \pi}{32} \right) = 10 + \frac{225 \pi}{32} \text{ m}^3/\text{min} \]

3. (3.10) A trough is 10 ft long and its ends have the shape of isosceles triangles that are 6 ft across the top and have a height of 4 ft. If the trough is filled with water at a rate of 12 ft³/min, how fast is the water level rising when the width of water across the top of the trough is 2 ft?

\[ \text{WANT: } \frac{dh}{dt} \text{ when } b = 2 \text{ ft} \]

\[ \text{Know: } \frac{dV}{dt} = 12 \text{ ft}^3/\text{min} \]

\[ V = \frac{1}{2} bh(10) = 5bh \]

\[ V = 5 \left( \frac{3}{2} h \right) h = \frac{15}{2} h^2 \]

\[ \frac{dV}{dt} = 15h \cdot \frac{dh}{dt} \]

When \( b = 2 \), what is \( h \)?

\[ \frac{a}{h} = \frac{3}{2} h \Rightarrow h = \frac{4}{3} \text{ ft} \]

So,

\[ 12 = 15 \left( \frac{4}{3} \right) \left( \frac{dh}{dt} \right) \]

\[ 12 = 20 \frac{dh}{dt} \]

\[ \frac{dh}{dt} = \frac{12}{20} = \frac{3}{5} \text{ ft/min} \]
4. (3.10) The right triangle below has the property that $x$ is decreasing at a rate of 2 in/sec.

(a) Suppose the area of the triangle remains a constant 24 in$^2$.
At what rate must $y$ be increasing when $x = 4$?

WANT: $\frac{dy}{dt}$ when $x = 4$

KNOW: $\frac{dx}{dt} = -2 \text{ in/sec}$ and $\frac{dA}{dt} = 0$ [A is constant]

$$A = \frac{1}{2}xy$$

$$\frac{dA}{dt} = \frac{1}{2} \frac{dx}{dt} \cdot y + \frac{1}{2}x \frac{dy}{dt}$$

When $x = 4$, since $A = 24$ always, then $24 = \frac{1}{2} (4)(y)$

$$12 = y$$

$$0 = \frac{1}{2} (-2)(12) + \frac{1}{2} (4) \cdot \frac{dy}{dt}$$

$$0 = -12 + 2 \frac{dy}{dt}$$

$$\frac{dy}{dt} = 6 \text{ in/sec}$$

(b) At what rate is the hypotenuse, $z$, changing at this same moment?

WANT: $\frac{dz}{dt}$ when $x = 4$.

KNOW: $\frac{dx}{dt} = -2 \text{ in/sec}$

$$\frac{dy}{dt} = 6 \text{ in/sec at this moment.}$$

$$x^2 + y^2 = z^2$$

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 2z \frac{dz}{dt}$$

When $x = 4$, we know $y = 12$, so $z^2 = 4^2 + 12^2 = 16 + 144 = 160$

$$z = \sqrt{160} \text{ in.}$$

$$4(-2) + 12(6) = \sqrt{160} \cdot \frac{dz}{dt}$$

$$64 = \sqrt{160} \cdot \frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{64}{\sqrt{160}} \text{ in/sec}$$
5. (3.11) Consider the function \( f(x) = 1 + x^2 \). Find \( \Delta y \) and \( dy \) when \( x = 4 \) and \( \Delta x = 0.25 \). What is the error in using differentials to approximate \( f(4.25) \)?

\[
\begin{align*}
\Delta y &= 19.0625 - 17 \\
\Delta y &= 2.0625
\end{align*}
\]

Error would be \( 0.0625 \)

\[
\begin{align*}
dy &= f'(4) \, dx \\
f'(x) &= 2x \\
f'(4) &= 8 \\
dy &= 8 \cdot dx = 8 \cdot (0.25) = 2 \\
dy &= 2
\end{align*}
\]

6. (3.11) Use differentials to find an approximate value for \( \sqrt[3]{26.9} \). Express your answer as an exact fraction.

\[
\sqrt[3]{26.9} = f(26.9) \approx f(27) + dy
\]

\[
\begin{align*}
\sqrt[3]{26.9} &= f(26.9) \approx f(27) + dy \\
f'(x) &= \frac{1}{3}x^{-2/3} \\
f'(x) &= \frac{1}{3 \cdot 27^{-2/3}} = \frac{1}{3} \cdot \frac{1}{27^{2/3}} = \frac{1}{3} \cdot \frac{1}{9} = \frac{1}{27}
\end{align*}
\]

\[
\sqrt[3]{26.9} \approx 3 + \frac{1}{27}(-0.1) = 3 + \frac{1}{27}(-\frac{1}{10}) = 3 - \frac{1}{270}
\]

\[
= \frac{809}{270}
\]

7. (3.11) Find the linearization of \( f(x) = \frac{1}{(x+1)^2} \) at \( a = 2 \) and use it to approximate \( \frac{1}{3.01^2} \). Express your answer as an exact fraction.

\[
L(x) = f(2) + f'(2)(x-2)
\]

\[
\begin{align*}
f(2) &= \frac{1}{9} \\
f'(2) &= \frac{-2}{27}
\end{align*}
\]

\[
L(x) = \frac{1}{9} - \frac{2}{27}(x-2)
\]

To approximate \( \frac{1}{(3.01)^2} \), what does \( x \) need to be?

\[
\frac{1}{(x+1)^2} = \frac{1}{(3.01)^2} \quad \text{when} \quad x+1 = 3.01
\]

\[
x = 2.01
\]

\[
L(2.01) = \frac{1}{9} - \frac{2}{27}(0.01)
\]

\[
= \frac{1}{9} - \frac{2}{27}(\frac{1}{100}) = \frac{1}{9} - \frac{1}{1350}
\]

\[
= \frac{149}{1350}
\]
8. (3.11) Find the quadratic approximation of \( f(x) = \cos x \) at \( a = \frac{\pi}{3} \).

\[
Q(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2} \left(x - \frac{\pi}{3}\right)^2
\]

\[
f(x) = \cos x \quad f\left(\frac{\pi}{3}\right) = \frac{1}{2}
\]

\[
f'(x) = -\sin x \quad f'\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}
\]

\[
f''(x) = -\cos x \quad f''\left(\frac{\pi}{3}\right) = -\frac{1}{2}
\]

\[
Q(x) = \frac{1}{2} - \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{3}\right) - \frac{1}{4} \left(x - \frac{\pi}{3}\right)^2
\]

9. (4.1) Calculate the following limits.

(a) \( \lim_{x \to -\infty} \pi^{-x} = \infty \)

Since \( \pi > 1 \), \( \pi^{-x} \) is a decreasing exponential. So as \( x \to -\infty \), \( \pi^{-x} \to \infty \).

(b) \( \lim_{z \to -2} \left( \frac{z+2}{z} \right)^{\frac{z+2}{z}} \to 0 \)

\( \lim_{x \to +2^-} \frac{x+3}{x-2} = -\infty \). Since \( \frac{5}{2} > 1 \), \( \frac{x-\infty}{\frac{5}{2}} \to 0 \)

(c) \( \lim_{x \to \infty} \frac{2e^{3x} - e^{-4x}}{3e^{3x} + 5e^{-4x}} = \lim_{x \to \infty} \frac{e^{3x} \left[ 2 - \frac{e^{-x}}{e^{3x}} \right]}{3e^{3x} + 5} = \frac{2}{3} \)

(d) \( \lim_{x \to \infty} \frac{2e^{3x} - e^{-4x}}{3e^{3x} + 5e^{-4x}} = \lim_{x \to \infty} \frac{e^{-4x} \left[ 2e^{7x} - \frac{1}{e^{4x}} \right]}{3e^{4x} + 5} = \frac{-1}{5} \)
10. (4.1) Differentiate the following. Do Not Simplify.

(a) \( f(x) = e^{\sqrt{\sin x}} \cos(3e^x) \)

\[ f'(x) = e^{\sqrt{\sin x}} \cdot \frac{\cos x}{2\sqrt{\sin x}} \cdot \cos(3e^x) + e^{\sqrt{\sin x}} (-\sin(3e^x) \cdot 3e^x) \]

(b) \( g(x) = \frac{e^{-5x}}{e + \tan(e^{x^2 - 7x})} \)

\[ g'(x) = \frac{\left[ e + \tan(e^{x^2 - 7x}) \right] (-5e^{-5x}) - e^{-5x} \left[ \sec^2(e^{x^2 - 7x}) e^{x^2 - 7x} (2x - 7) \right]}{(e + \tan(e^{x^2 - 7x}))^2} \]

11. (4.1) Find \( f^{(n)}(x) \) for \( f(x) = e^{-3x} \)

\[ f'(x) = 3e^{-3x} \]
\[ f''(x) = -3^2 e^{-3x} \]
\[ f'''(x) = -3^3 e^{-3x} \]
\[ f^{(n)}(x) = -3^n e^{-3x} \]

\[ f^{(n)}(x) = (-1)^n \cdot 3^n \cdot e^{-3x} \]

12. (4.1) Find the slope of the tangent line to the curve \( e^{x^2y} + \cos(e^y) = 2x^2 - y \) at the point \( (1, 0) \).

\[ e^{x^2y} \left[ 2xy + x^2 \cdot \frac{dy}{dx} \right] - \sin(e^{y-1})e^y \frac{dy}{dx} = 4x - \frac{dy}{dx} \]

At \((1, 0)\): \( e^0 \left[ 0 + \frac{dy}{dx} \right] - \sin(e^{0-1})e^0 \frac{dy}{dx} = 4 - \frac{dy}{dx} \)

\[ \frac{dy}{dx} = 4 - \frac{dy}{dx} \]

\[ \sin(e^{0-1}) = \sin(1-1) = \sin 0 = 0 \]

Also, recall \( e^0 = 1 \).

\[ \frac{dy}{dx} = 4 \]

\[ \frac{dy}{dx} = 2 \]