Homework 2: Sections 1.3, 2.2, and 2.3

Show ALL work on your own paper to get full credit. This assignment is due at the beginning of class on Friday, February 4.

A portion of the following problems will be graded for content. The remaining problems will be spot-checked for completion and work shown.

1. (1.3) Eliminate the parameter to find a Cartesian equation of the curve. Sketch the curve and describe how it is traced out as the parameter value increases.

\[ x = -2 - t, \quad y = -1 - t^2, \quad -2 \leq t < 1 \]

\[
\begin{align*}
x + 2 &= -t \\
-(x + 2) &= t \\
y &= -1 - (-t)^2 \\
y &= -1 - (x + 2)^2
\end{align*}
\]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( X )</th>
<th>( Y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>-3</td>
<td>-2</td>
</tr>
</tbody>
</table>

Direction is from right to left along parabola.

2. (1.3) Find a Cartesian equation and sketch the curve defined by the vector function \( \mathbf{r}(t) = \langle 2 \cos t + 1, -2 \sin t - 1 \rangle \). How is the curve traced out as \( t \) increases?

\[
\begin{align*}
x &= 2 \cos t + 1 \\
y &= -2 \sin t - 1 \\
\frac{x - 1}{2} &= \cos t \\
\frac{y + 1}{2} &= \sin t \\
\sin^2 t + \cos^2 t &= 1 \\
\left(\frac{y + 1}{2}\right)^2 + \left(\frac{x - 1}{2}\right)^2 &= 1 \\
\frac{(y + 1)^2}{4} + \frac{(x - 1)^2}{4} &= 1 \\
(x - 1)^2 + (y + 1)^2 &= 4
\end{align*}
\]

<table>
<thead>
<tr>
<th>( t )</th>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>-1</td>
</tr>
<tr>
<td>( \pi/2 )</td>
<td>1</td>
<td>-3</td>
</tr>
</tbody>
</table>

Circle centered at \((1, -1)\) with radius 2.
Direction is clockwise.
3. (1.3) Find parametric equations for the line that passes through the point \((-3, 4)\) and is parallel to the line \(y = \frac{2}{5}x + 2\).

\[
\vec{r}_0 = \langle -3, 4 \rangle
\]

Slope is \(\frac{2}{5}\), so a vector parallel to line is \(\vec{v} = \langle 5, 2 \rangle\)

\[
\vec{r}(t) = \vec{r}_0 + t\vec{v} = \langle -3, 4 \rangle + t\langle 5, 2 \rangle
\]

\[
= \langle -3 + 5t, 4 + 2t \rangle
\]

\[
\begin{align*}
x &= -3 + 5t \\
y &= 4 + 2t
\end{align*}
\]

4. (1.3) A line is parametrically defined by \(x = 7 + 9t, \ y = 2 - 4t\). (a) What is the slope of this line? (b) Find a vector perpendicular to this line.

\[
x = 7 + 9t \quad \text{\textcircled{1}}
\]

\[
y = 2 - 4t \quad \text{\textcircled{2}}
\]

\((7, 2)\) is a point on line

\(<9, -4>\) is a vector parallel to line.

Slope is \(\frac{-4}{9}\)

A vector perpendicular to the line is \(\langle 4, 9 \rangle\)

5. (2.2) Problem #4 from Section 2.2 in your textbook. (You do not need to redraw the graph.)

(a) \(\lim_{x \to 1} g(x) = 0\)

(b) \(\lim_{x \to 0} g(x) \) DNE since \(\text{LHL} = \infty, \ \text{RHL} = -\infty\)

(c) \(\lim_{x \to 2} g(x) = 1\)

(d) \(\lim_{x \to -2} g(x) = 0\)

(e) \(\lim_{x \to -1} g(x) = 1\)

(f) \(\lim_{x \to -1} g(x) \) DNE since \(\text{LHL} = 1, \ \text{RHL} = 0\).
6. (2.2) Problem #8 from Section 2.2 in your textbook.

(i) \( \lim_{x \to -1^-} g(x) = 3 \)

(ii) \( \lim_{x \to -1^+} g(x) = -1 \)

(iii) \( \lim_{x \to -1} g(x) \) DNE (since LHL \( \neq \) RHL)

(iv) \( \lim_{x \to 1^-} g(x) = 1 \)

(v) \( \lim_{x \to 1^+} g(x) = 3 \)

(vi) \( \lim_{x \to 1} g(x) \) DNE (since LHL \( \neq \) RHL)

7. (2.2) Find all vertical asymptotes of the function \( f(x) = \frac{(x^2 + 4x + 3)(x - 2)}{(x^2 - 3x + 2)(x + 3)^2} \)

\[
f(x) = \frac{(x+1)(x+3)(x-2)}{(x-2)(x-1)(x+3)^2} = \frac{(x+1)}{(x-1)(x+3)}
\]

\( \text{VA: } x = 1, x = -3 \)

8. (2.2) Calculate the following limits (if they exist).

(a) \( \lim_{x \to -2^-} \frac{x(x-3)^2}{(x-4)(x-2)} = \infty \)

\( \text{Signs: } \frac{(+)(-)^2}{(-)(-)} = \frac{(+)(+)}{(-)(-)} = + \)

(b) \( \lim_{x \to -2^+} \frac{x(x-3)^2}{(x-4)(x-2)} = -\infty \)

\( \text{Signs: } \frac{(+)(-)^2}{(-)(+)} = \frac{(+)(+)}{(-)(+)} = - \)

(c) \( \lim_{x \to -4} \frac{(x+6)(x-5)^5}{(x+4)^2} = \infty \)

\( \text{Note: Since } x+4 \text{ term is squared, limits from left and right will be same.} \)

\( \text{Signs: } \frac{(+)(-)^5}{(+)} = \frac{(+)(-)}{(+) = -} \)
9. (2.3) Calculate \( \lim_{x \to 5} \frac{x - 5}{5 - x} \):
\[
\lim_{x \to 5} \frac{x - 5}{5 - x} = \lim_{x \to 5} \frac{x - 5}{5x(5-x)} = \lim_{x \to 5} \left( \frac{1}{5x} \right) = \left( \frac{1}{25} \right)
\]

10. (2.3) Finish the following problem from the 2.3 lecture notes:

Calculate \( \lim_{t \to 3} \mathbf{r}(t) \), where \( \mathbf{r}(t) = \left< \frac{2}{t-3} - \frac{12}{t^2-9}, \frac{t-3}{\sqrt{t^2+7} - 4} \right> \):

\[
\lim_{t \to 3} \left[ \frac{2}{t-3} - \frac{12}{t^2-9} \right] = \lim_{t \to 3} \left[ \frac{2(t+3) - 12}{(t-3)(t+3)} \right] = \lim_{t \to 3} \left[ \frac{2(t-3)}{(t-3)(t+3)} \right] = \frac{2}{6} = \frac{1}{3}
\]

\[
\lim_{t \to 3} \frac{t-3}{\sqrt{t^2+7} - 4} = \lim_{t \to 3} \frac{t-3}{\sqrt{t^2+7} + 4} \cdot \frac{\sqrt{t^2+7} + 4}{\sqrt{t^2+7} - 4} = \lim_{t \to 3} \frac{(t-3)(\sqrt{t^2+7} + 4)}{t^2+7-16} = \frac{8}{6} = \frac{4}{3}
\]

So, \( \lim_{t \to 3} \mathbf{r}(t) = \left< \frac{1}{3}, \frac{4}{3} \right> \)
11. (2.3) Calculate \( \lim_{x \to -7} \frac{x^2 + 6x - 7}{|x + 7|} \) and \( \lim_{x \to -7^+} \frac{x^2 + 6x - 7}{|x + 7|} \).

\[
\lim_{x \to -7^-} \frac{(x+7)(x-1)}{|x+7|} = \lim_{x \to -7^-} \frac{(x+7)(x-1)}{-(x+7)} = \lim_{x \to -7^-} -(x-1) = 8
\]

\[
\lim_{x \to -7^+} \frac{(x+7)(x-1)}{|x+7|} = \lim_{x \to -7^+} \frac{(x+7)(x-1)}{x+7} = \lim_{x \to -7^+} (x-1) = -8
\]

12. (2.3) Show that \( \lim_{x \to 0} x^4 \cos \left( \frac{2}{x} \right) = 0 \) by using the Squeeze Theorem.

Know 

\[-1 \leq \cos \left( \frac{2}{x} \right) \leq 1 \quad \text{for all } x \neq 0
\]

\[-x^4 \leq x^4 \cos \left( \frac{2}{x} \right) \leq x^4 \quad \text{for all } x \neq 0.
\]

Since \( \lim_{x \to 0} (-x^4) = 0 \) and \( \lim_{x \to 0} x^4 = 0 \), then by Squeeze Theorem, \( \lim_{x \to 0} x^4 \cos \left( \frac{2}{x} \right) = 0. \)
Bonus: Find numbers $a$ and $b$ such that $\lim_{x \to 0} \frac{\sqrt{ax+b} - 2}{x} = 1$. (Show how you get your answers.)

Form of limit: $\frac{0+b-2}{0} = \frac{\sqrt{b} - 2}{0}$

In order for limit to equal 1 and not $+\infty$ or $-\infty$, we need it to have form $\frac{0}{0}$ and not $\frac{\text{nonzero}}{0}$.

So, $\sqrt{b} - 2 = 0$

$$\sqrt{b} = 2$$

$$b = 4$$

Now, $\lim_{x \to 0} \frac{\sqrt{ax+b} + 2}{\sqrt{ax+b} + 2} = \lim_{x \to 0} \frac{ax + 4 - 4}{x(\sqrt{ax+b} + 2)}$

$$= \lim_{x \to 0} \frac{ax}{x(\sqrt{ax+b} + 2)} \cdot \lim_{x \to 0} \frac{a}{\sqrt{ax+b} + 2} = \frac{a}{\sqrt{a+2}} = \frac{a}{4}$$

We want limit to be 1, so

$$\frac{a}{4} = 1$$

$$a = 4$$