6.3 The Definite Integral

We saw in the previous section that if \( f(x) \geq 0 \) on an interval \([a, b]\), then the exact area under the graph between \( x = a \) and \( x = b \) is

\[
A = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i
\]

We now define the **definite integral of \( f \) from \( a \) to \( b \)** as the above limit:

\[
\int_{a}^{b} f(x) \, dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i
\]

If the subintervals are equally spaced, then \( \Delta x_i = \frac{b - a}{n} \) and this limit can be represented as:

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x_i
\]

If the limit exists, then \( f \) is said to be **integrable** on this interval.

Notes: In the notation \( \int_{a}^{b} f(x) \, dx \), \( f(x) \) is called the **integrand** and \( a \) and \( b \) are call the **limits of integration**. (\( a \) is the lower limit and \( b \) is the upper limit).

Once again, if \( f(x) \geq 0 \) on the interval, then the definite integral can be interpreted as the area under the graph of \( f \) between \( x = a \) and \( x = b \)

*If \( f \) is not always positive, the definite integral is still defined, but now represents the **NET** area.*

For the graph of \( f \) below, compute the following definite integrals using the indicated areas.

\[
\int_{0}^{A} f(x) \, dx =
\]

\[
\int_{A}^{B} f(x) \, dx =
\]

\[
\int_{A}^{C} f(x) \, dx =
\]

\[
\int_{0}^{C} f(x) \, dx =
\]

\[
\int_{0}^{D} f(x) \, dx =
\]

\[
\int_{C}^{D} f(x) \, dx =
\]
The **Midpoint Rule** for definite integrals means to approximate the integral by using a midpoint Riemann Sum (just as in 6.2).

Use the Midpoint Rule with $n = 4$ to approximate $\int_{-4}^{4} (x^2 - 4) \, dx$.

Evaluate the following definite integrals by interpreting each in terms of area.

(1) $\int_{-1}^{6} (2x - 4) \, dx$

(2) $\int_{0}^{4} \sqrt{16 - x^2} \, dx$
\[\int_{-2}^{5} |x - 3| \, dx\]

Properties of the Integral:

1. \[\int_{a}^{b} c \, dx = c(b - a)\] where \(c\) is any constant

2. \[\int_{a}^{b} [f(x) \pm g(x)] \, dx = \int_{a}^{b} f(x) \, dx \pm \int_{a}^{b} g(x) \, dx\]

3. \[\int_{a}^{b} cf(x) \, dx = c \int_{a}^{b} f(x) \, dx\]

4. \[\int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx = \int_{a}^{b} f(x) \, dx\]

5. \[\int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx\]

6. \[\int_{a}^{a} f(x) \, dx = 0\]

Example: If \[\int_{2}^{5} f(x) \, dx = 7\] and \[\int_{2}^{5} g(x) \, dx = -4\], calculate \[\int_{2}^{5} (3f(x) + g(x)) \, dx\].
Example: Write the following as a single integral. \[ \int_{-2}^{3} f(x) \, dx - \int_{-2}^{0} f(x) \, dx + \int_{3}^{5} f(x) \, dx \]

More Properties:

1. If \( f(x) \geq 0 \) for all \( x \) on \([a, b]\), then \( \int_{a}^{b} f(x) \, dx \geq 0 \)

2. If \( f(x) \geq g(x) \) for all \( x \) on \([a, b]\), then \( \int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} g(x) \, dx \)

3. If \( m \leq f(x) \leq M \) for all \( x \) on \([a, b]\), then

\[
m(b - a) \leq \int_{a}^{b} f(x) \, dx \leq M(b - a)
\]

Estimate the value of the following integral by finding an upper and lower bound and applying Property 3 above.

\[ \int_{1}^{3} \sqrt{1 + x^3} \, dx \]
6.4 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, Part 1:

If \( f \) is continuous on \([a, b]\), then the function \( g(x) \) defined by

\[
  g(x) = \int_a^x f(t) \, dt \quad a \leq x \leq b
\]

is continuous on \([a, b]\) and differentiable on \((a, b)\) and

\[
  g'(x) = f(x)
\]

In other words,

\[
  \frac{d}{dx} \int_a^x f(t) \, dt = f(x)
\]

Let \( g(x) = \int_0^x f(t) \, dt \), where \( f \) is graphed below. Calculate \( g(0), g(2), g(6) \).

\( g \) represents the “area so far”. The FTC Part 1 says that \( g \) is an antiderivative of \( f \) since \( g'(x) = f(x) \).

Calculate the derivatives of the following functions.

1. \( g(x) = \int_{-7}^x \sqrt[3]{6 - t^2} \, dt \)

2. \( g(x) = \int_3^{x^2} \frac{\ln u}{u^2 + 3} \, du \)

3. \( g(x) = \int_{\sqrt{x}}^{\cos x} \frac{\tan t}{t^2} \, dt \)
The Fundamental Theorem of Calculus, Part 2:

If $f$ is continuous on $[a, b]$, then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$.

Evaluate the following definite integrals.

$$\int_{1}^{3} \left( \frac{1}{x^3} - \frac{3}{x} \right) \, dx$$

$$\int_{0}^{2} (3e^x - 2^x) \, dx$$

$$\int_{1}^{4} \frac{u^3 - 1}{\sqrt{u}} \, du$$

$$\int_{-2}^{0} (t^2 - t)^2 \, dt$$
\[ \int_0^\pi (\sec^2 \theta + 2 \cos \theta) \, d\theta \]

Find the area under the curve \( f(x) = |x^2 - 4| \) between \( x = 0 \) and \( x = 4 \).

The **indefinite integral** is used to indicate the process of finding the most general antiderivative of \( f(x) \):

\[ \int f(x) \, dx = F(x) + C \]

where \( F(x) \) is an antiderivative of \( f(x) \) (i.e. \( F'(x) = f(x) \)).

Find the general indefinite integral \( \int \left[ \frac{5}{1 + x^2} + (x - 1)(2x + 3) \right] \, dx \)
Application: Suppose an object has velocity function \( v(t) \). Then, on the time interval from \( t = a \) to \( t = b \):

\[
\int_a^b v(t) \, dt = s(b) - s(a)
\]

by the 2nd Part of the Fundamental Theorem since the position function \( s(t) \) is an antiderivative of \( v(t) \). This is the \textit{displacement} of the object on the time interval.

The actual total distance traveled by the object is \( \int_a^b |v(t)| \, dt \). Why?

Example: Suppose the velocity of an object is given by the function \( v(t) = t^2 - 2t - 8 \). Find the displacement and total distance traveled by the object on the time interval \( 0 \leq t \leq 6 \).