5.7 Applied Maximum/Minimum Problems

Finding extreme values of functions is useful in applications.

When dealing with a max/min problem, recognize what you are trying to maximize/minimize, write this quantity in terms of only ONE variable, and use calculus to find the max/min value of this function. You must show that your answer is a max or min.

A rectangular storage container with an open top is to have a volume of 10 m³. If the length of the base is three times the width, find the dimensions of the container that will minimize the cost of materials (i.e. minimize the surface area).
Find the dimensions of the largest rectangle that has its base on the $x$-axis and its other two vertices above the $x$-axis lying on the function $f(x) = 11 - x^4$. 
Find the point on the line $y = 5x + 2$ that is closest to the point $(4,1)$. 
A company is designing a closed cylindrical can in which the top and bottom of the can cost $3 per cm$^2$ and the side of the can costs $2 per cm$^2$. If the company can afford to spend $50 per can, determine the dimensions of the can that would maximize its volume.
5.7 Antiderivatives

Given a function, you are able to take its derivative. But now, we will talk about how to find the original function given its derivative.

Definition: A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

Find an antiderivative of $f(x) = 3x^2$. How many are there?

If $F$ is an antiderivative of $f$, then the most general antiderivative of $f$ is $F(x) + C$, where $C$ is an arbitrary constant.

What is the general antiderivative of $f(x) = x^n$ if $n \neq -1$?

Table of Some Antiderivatives:

<table>
<thead>
<tr>
<th>Function</th>
<th>General Antiderivative</th>
<th>Function</th>
<th>General Antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$ (a constant)</td>
<td>$ax + C$</td>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
<td>$\arcsin x + C$</td>
</tr>
<tr>
<td>$x^n$, $n \neq -1$</td>
<td>$\frac{x^{n+1}}{n+1} + C$</td>
<td>$\frac{1}{1+x^2}$</td>
<td>$\arctan x + C$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\ln</td>
<td>x</td>
<td>+ C$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x + C$</td>
<td>$\cos x$</td>
<td>$\sin x + C$</td>
</tr>
<tr>
<td>$a^x$</td>
<td>$\frac{a^x}{\ln a} + C$</td>
<td>$\sec^2 x$</td>
<td>$\tan x + C$</td>
</tr>
<tr>
<td>$\frac{1}{x}$</td>
<td>$\frac{1}{x}$</td>
<td>$\sec x \tan x$</td>
<td>$\sec x + C$</td>
</tr>
<tr>
<td>$\csc x \cot x$</td>
<td>$-\csc x + C$</td>
<td>$\csc^2 x$</td>
<td>$-\cot x + C$</td>
</tr>
</tbody>
</table>

Find the most general antiderivative of $f(x) = 7x^5 + \sqrt{x} + \frac{3}{\sqrt{x}} + 2\cos x + \frac{3}{\sqrt{x}^3} + \frac{9}{1+x^2}$.

Find $f(x)$ given that $f'(x) = \frac{1}{\sqrt{1-x^2}} - \sin x + 3 + \sec^2 x$ and $f(0) = -8$.
Find $f(x)$ given that $f'(x) = 10 - \frac{2}{x^3} - \frac{6}{x^2} + \frac{3}{x}$ and $f(1) = 7$.

Find $f(x)$ given that $f''(x) = \frac{x + \sqrt{x}}{\sqrt{x}} + e^x$ along with the data $f'(1) = \frac{5}{3}$ and $f(0) = -1$.

Find $f(x)$ given that $f''(x) = \frac{1}{4x\sqrt{x}}$ where $f(1) = 4$ and $f(4) = 12$. 
Application: If we know the velocity function \( v(t) \) of an object, we can now find the position function \( s(t) \), since \( v(t) = s'(t) \). Similarly, given the acceleration function \( a(t) \), we can find both the velocity and position functions.

Example: A particle moving in a straight line has acceleration function \( a(t) = 10 + 3t - t^2 \). It is known that its initial velocity is 2 m/s and its position at time \( t = 0 \) is 10 m. Find the position function.

Example: An object is thrown upward with a speed of 5 m/s from a 400 m observation tower. Find the distance of the stone above the ground at time \( t \).

Fact: Ignoring air resistance, the acceleration due to gravity is \(-9.8 \text{ m/s}^2\) (or \(-32 \text{ ft/s}^2\)).