3.5 The Chain Rule

To find the derivative of a function that is the composition of two functions for which we already know the derivatives, we can use the Chain Rule.

The Chain Rule: Suppose $F(x) = f(g(x))$. Then, provided that $g'(x)$ and $f'(g(x))$ both exist,

$$F'(x) = f'(g(x))g'(x)$$

Alternate way of thinking about it: If $y = f(u)$ and $u = g(x)$ where both are differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

Find the derivatives of the following functions.

- $f(x) = \sqrt{x^2 + 9 + \cos(x^3)}$

- $f(x) = \left(\frac{5x - 7}{4x^3 - x}\right)^{10}$

- $g(t) = \left(3t^2 + \frac{2}{t^4}\right)^5 (2t - \cos t)^7$

- $g(x) = \sqrt[4]{5x^4 + \sqrt{6x} + 1}$
\begin{itemize}
  \item $f(x) = \tan 5x \sec 3x$
  
  \item $g(x) = \sin^2 x + \cos^4 2x$
  
  \item $h(x) = \sec(\sin x)$
  
  \item $f(x) = \tan^3(csc 4x)$
  
  \item Suppose $f'(4) = 3, f'(5) = 2, g'(4) = 6$ and $g(4) = 5$.
    
    (a) Calculate $F'(4)$ where $F(x) = f(g(x))$.

    (b) Calculate $G'(0)$ where $G(x) = [f(5 + \sin 3x)]^2$
\end{itemize}
3.6 Implicit Differentiation

Consider the equation $y^3 + 2xy = 9$. This equation cannot easily be solved for $y$. So how can we find $y'$?

When $y$ cannot be written explicitly as a function of $x$ (or not easily), we can use the method of **implicit differentiation**.

To find $\frac{dy}{dx}$, differentiate both sides with respect to $x$, remembering that the Chain Rule is necessary since $y$ is dependent on $x$.

Find $\frac{dy}{dx}$ for the equation $x^2 + y + 3y^4 = 16$

Find $y'$ if $(y^2 + 1)^3 + xy = 3x^2 - 2y$. What is the slope of the tangent line at the point $(2,1)$?

Find $\frac{dy}{dx}$ where $\sin(x\sqrt{y}) = \cos 2y - y \cos x$. 

Find $\frac{dy}{dx}$ where $\sin(x\sqrt{y}) = \cos 2y - y \cos x$. 

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Find \( \frac{dx}{dy} \) if \( x^2 - 5x^4y^2 = 4y^3 \)

Example: Find an equation of the tangent line to the hyperbola \( \frac{y^2}{36} - \frac{x^2}{4} = 1 \) at the point \((1, -3\sqrt{5})\).

The curves \( x^2 - y^2 = 5 \) and \( 4x^2 + 9y^2 = 72 \) intersect at the point \((3, -2)\). Show that the tangent lines to the two curves at this point are orthogonal.
3.7 Derivatives of Vector Functions

If \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) is a vector function, then \( \mathbf{r}'(t) = \langle x'(t), y'(t) \rangle \) if both of these derivatives exist.

The derivative at \( a \), \( \mathbf{r}'(a) \), is a tangent vector to the curve when \( t = a \). In the context of motion, it represents the instantaneous velocity of an object with position function \( \mathbf{r}(t) \). So, the tangent vector, (or velocity vector), points in the direction of motion as \( t \) increases.

Example: Find the domain of the vector function \( \mathbf{r}(t) = \left\langle \frac{t}{t-6}, \sqrt{3t-5} \right\rangle \).

Find the derivative of the vector function and its domain.

Find a tangent vector to the curve at the point where \( t = 3 \).

Find parametric equations for the tangent line to the curve at this point.
Example: Find a unit tangent vector to the curve $\mathbf{r}(t) = \langle t \sin t, 4 - 2 \cos 3t \rangle$ at the point where $t = \frac{\pi}{2}$.

Find a unit tangent vector to the curve $\mathbf{r}(t) = \langle 5t^2 + 1, 8t^2 - t \rangle$ at the point $(6, 9)$. 
Again, if \( \mathbf{r}(t) = \langle x(t), y(t) \rangle \) is vector function representing the position of a particle at time \( t \), then the tangent vector \( \mathbf{r}'(t) \) is the instantaneous velocity at time \( t \) and the instantaneous speed at time \( t \) is \( |\mathbf{r}'(t)| \).

(Velocity = Vector; Speed = Scalar)

Example: A projectile is fired so that its position is given by the function \( \mathbf{r}(t) = \langle 48\sqrt{3}t, 48t - 16t^2 \rangle \).

Find the velocity and speed at time \( t = 2 \).

What is the velocity when the projectile hits the ground?

To find the angle of intersection between two curves, find the angle between the tangent vectors at the point of intersection.

Example: Find the angle of intersection of the curves \( \mathbf{r}_1(t) = \langle 4 - t, t^2 - 5 \rangle \) and \( \mathbf{r}_2(s) = \langle \sqrt{s - 1}, s + 2 \rangle \) if it is known the curves intersect at the point \( (1, 4) \).