2.5 Continuity

In Section 2.3 we saw that the limit as \( x \) approaches \( a \) can sometimes be found by evaluating the function at \( a \). If this is the case, then the function is continuous.

Definition: A function is \textit{continuous} at a number \( a \) if

\[
\lim_{x \to a} f(x) = f(a)
\]

Otherwise, we say the function is \textit{discontinuous} at \( a \), or that there is a discontinuity at \( a \).

In order for a function to be continuous at a number \( a \):

1. \( f(a) \) must be defined.
2. \( \lim_{x \to a} f(x) \) must exist. (The left-handed and right-handed limits must both equal the same value.)
3. \( \lim_{x \to a} f(x) = f(a) \)

Examples of discontinuities: Holes, vertical asymptotes, and jumps.

A “hole” in a graph is also referred to as a \textit{removable discontinuity} because if we wanted to, we could just redefine the function at that point to make it continuous.

Removable discontinuities occur where the limit exists at \( a \) (left and right limits are equal), but is not equal to \( f(a) \).

A vertical asymptote is referred to as an \textit{infinite discontinuity}.

A jump in the graph is referred to as a \textit{jump discontinuity}.

Jumps occur where the limits from the left and right exist, but are not equal.
A function is **continuous from the left** at a number \( a \) if \( \lim_{x \to a^-} f(x) = f(a) \) and **continuous from the right** if \( \lim_{x \to a^+} f(x) = f(a) \).

Examples: Determine where the functions below are discontinuous. State the type of discontinuity and explain why mathematically using limits. Is the function continuous from the left or right there?

1. \( f(x) = \begin{cases} 
\frac{(x+7)(x-5)}{x-5} & \text{if } x \neq 5 \\
6 & \text{if } x = 5 
\end{cases} \)

2. \( f(x) = \begin{cases} 
x^2 - 4 & \text{if } x \leq -1 \\
x + 1 & \text{if } x > -1 
\end{cases} \)

3. \( f(x) = \frac{(x+3)(x-2)}{(x-2)^3} \)

Fact: All polynomials are continuous everywhere!

Fact: A rational function is continuous wherever it is defined! (Remember if the denominator is 0, you will have an infinite discontinuity (vertical asymptote) or a removable discontinuity (hole).)
(4) \( f(x) = \begin{cases} 
3x + 1 & \text{if } x < -2 \\
x^2 - 7 & \text{if } -2 \leq x \leq 3 \\
\frac{x - 2}{x^3 - 25} & \text{if } x > 3 
\end{cases} \)

For what value of \( c \), if any, is the following function continuous?

\( f(x) = \begin{cases} 
x^2 + c^2 & \text{if } x \leq -4 \\
cx + 12 & \text{if } x > -4 
\end{cases} \)

What value of \( c \) would work if the function was defined as:

\( f(x) = \begin{cases} 
x^2 + c^2 & \text{if } x < -4 \\
14 & \text{if } x = -4 \\
cx + 12 & \text{if } x > -4 
\end{cases} \)
If \( f \) and \( g \) are continuous at \( a \) and \( c \) is any constant, then the functions \( f + g, f - g, cf, fg, \) and \( \frac{f}{g} \) (where \( g(a) \neq 0 \)) are all continuous functions.

The Intermediate Value Theorem: Suppose \( f \) is continuous on the closed interval \([a, b]\) and let \( N \) be any number strictly between \( f(a) \) and \( f(b) \). Then there exists a number \( c \) in \((a, b)\) such that \( f(c) = N \).

Example: If \( f(x) = x^4 - x^3 + 3x^2 + 2 \), show that there is a number \( c \) so that \( f(c) = 3 \).

Example: Show that the equation \( x^3 - 2x - 2 = 0 \) has a root on the interval \((1, 2)\).
2.6 Limits at Infinity; Horizontal Asymptotes

Up to this point, we have dealt with limits as $x$ approaches some number $a$. Now, we examine limits as $x$ approaches $\infty$ or $-\infty$. These are called limits at infinity.

Let $f$ be a function defined on some interval $(a, \infty)$. Then,

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to $L$ by taking $x$ sufficiently large.

**Definition:** If $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then $f(x)$ has a **horizontal asymptote** at $y = L$.

$$\lim_{x \to \infty} \frac{1}{x} = \lim_{x \to -\infty} \frac{2}{x} = \lim_{x \to \pm\infty} \frac{3}{x^2} = \lim_{x \to \infty} x =$$

$$\lim_{x \to -\infty} x = \lim_{x \to -\infty} x^2 = \lim_{x \to -\infty} x^2 =$$

$$\lim_{x \to \infty} (x^3 - x^2) =$$

$$\lim_{x \to \infty} \frac{x^3}{x^2} = \lim_{x \to \infty} \frac{x^2}{x^2} = \lim_{x \to \infty} \frac{x^2}{x^2} =$$

Evaluate the following limits:

$$\lim_{x \to \infty} \frac{2x^3 + x^2 - 1}{5x^3 - 7x + 2} = \lim_{x \to -\infty} \frac{2x^3 + x^2 - 1}{5x^3 - 7x + 2} =$$
\[ \lim_{x \to \infty} \frac{4x^2 - 2x + 3}{5 - 3x} \]

\[ \lim_{x \to -\infty} \frac{4x^2 - 2x + 3}{5 - 3x} \]

\[ \lim_{x \to \infty} \frac{x - 8}{2x^2 + x - 9} \]

\[ \lim_{x \to -\infty} \frac{x - 8}{2x^2 + x - 9} \]

\[ \lim_{x \to \infty} \frac{\sqrt{9x^2 - 12}}{5x + 2} \]

\[ \lim_{x \to -\infty} \frac{\sqrt{9x^2 - 12}}{5x + 2} \]
\[
\lim_{x \to \infty} \sqrt{x^2 - x - x}
\]

\[
\lim_{x \to -\infty} \sqrt{x^2 + 5x + x}
\]

Find all horizontal asymptotes of the function \( f(x) = \frac{4x - 5}{\sqrt{25x^2 + 1}} \).