2.4 The Precise Definition of a Limit

Reminders/Remarks: \(|x - 4| < 3\) means that the distance between \(x\) and 4 is less than 3. In other words, \(x\) lies strictly between 1 and 7. So, \(|x - a| < \delta\) means that the distance between \(x\) and \(a\) is less than \(\delta\).

Note: The inequality \(0 < |x - a| < \delta\) means the distance between \(x\) and \(a\) is less than \(\delta\), but \(x \neq a\).

Similarly, \(|f(x) - 4| < 3\) means that the distance between \(f(x)\) and 4 is less than 3. In other words, \(f(x)\) lies strictly between 1 and 7. So, \(|f(x) - L| < \epsilon\) means that the distance between \(f(x)\) and \(L\) is less than \(\epsilon\).

Consider the function \(f(x) = 2x + 4\). We know \(\lim_{x \to 3} (2x + 4) = 10\). We said that the limit means we can make \(f(x)\) as close to 10 as we want by getting \(x\) closer and closer to 3.

Question: How close to 3 does \(x\) need to be so that \(f(x)\) differs from 10 by less than 1?

Question: How close to 3 does \(x\) need to be so that \(f(x)\) differs from 10 by less than 0.1?

Question: How close to 3 does \(x\) need to be so that \(f(x)\) differs from 10 by less than an arbitrary number \(\epsilon\)?

This number that we’re finding is usually denoted as \(\delta\).
**Definition of a Limit:** Let $f$ be a function defined on some open interval that contains the number $a$, except possibly at $a$ itself. Then we say that the limit of $f(x)$ as $x$ approaches $a$ is $L$, and we write

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

To prove a limit using the definition of the limit, there are two steps:

1. Do some scratchwork to determine a value for $\delta$.
2. Show that this $\delta$ works.

Example: Prove using the definition of the limit that $\lim_{x \to 3}(2x + 4) = 10$. 


Example: Prove using the definition of the limit that \( \lim_{x \to -2} (3x + 7) = 1. \)

Consider the function \( f(x) = x^2 \). We know \( \lim_{x \to 2} x^2 = 4. \)

Question: Given \( \epsilon = 1 \), find a number \( \delta \) so that if \( 0 < |x - 2| < \delta \), then \( |x^2 - 4| < 1. \)

Question: Given \( \epsilon = 0.1 \), find a number \( \delta \) so that if \( 0 < |x - 2| < \delta \), then \( |x^2 - 4| < 0.1. \)
Consider the function $f(x) = \frac{1}{x^2}$. We know $\lim_{x \to 0} \frac{1}{x^2} = \infty$. We said that this means we can make $f(x)$ as large as we want by getting $x$ closer and closer to 0.

**Question:** Find a number $\delta$ so that if $0 < |x| < \delta$, then $\frac{1}{x^2} > 100$.

**Question:** Find a number $\delta$ so that if $0 < |x| < \delta$, then $\frac{1}{x^2} > 10,000$.

**Question:** Find a number $\delta$ so that if $0 < |x| < \delta$, then $\frac{1}{x^2} > M$, where $M$ is some arbitrary number.

**Definition:** Left $f$ be a function defined on some open interval that contains the number $a$, except possibly at $a$ itself. Then

$$\lim_{x \to a} f(x) = \infty$$

if for every number $M > 0$ there exists a corresponding number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$. 