1.3 Vector Functions

**Parametric Curves:** Sometimes, instead of representing a curve using just $x$ and $y$, it is more convenient to use **parametric equations** using a parameter such as $t$. This means that the values of $x$ and $y$ are defined as functions of this parameter, $x(t)$ and $y(t)$.

Example: Sketch the curve represented by the parametric equations $x = -3t + 1$, $y = 2t - 1$.

If given a set of parametric equations, it may be useful to convert back into a Cartesian equation (using $x$ and $y$ only). In order to do this, you must eliminate the parameter. How?

1. If possible, solve one of the parametric equations for $t$ and use substitution.

2. If the parametric equations involve trig functions, use a trig identity, often $\sin^2 \theta + \cos^2 \theta = 1$.

Example: Eliminate the parameter from the previous example and write a Cartesian equation for the curve.

Sometimes, there may be a restriction on the values of $t$ or the values of $x$ and $y$ may have bounds you need to watch out for.

If in the previous example, $x = -3t + 1$, $y = 2t - 1$, we also had the restriction $-1 \leq t < 2$, what does the curve look like?
Example: Eliminate the parameter to find a Cartesian equation for the following curves, sketch a graph, and describe the direction of motion as $t$ increases.

(a) $x = \sqrt{t}, \quad y = 2 - t$

(b) $x = 3 + \sin \theta, \quad y = 2 + \cos \theta$

**Vector Functions:** We can define vector functions using these parametric equations by $\mathbf{r}(t) = <x(t), y(t)>$. It is called a **vector function** because it takes values of $t$ and produces vectors. These vectors are tracing out the curve.
Example: Sketch the curve represented by the vector function \( \mathbf{r}(t) = (4 \cos t) \mathbf{i} + (\sin t) \mathbf{j} \), \( 0 \leq t \leq \pi \).

Example: The position of an object after \( t \) seconds is modeled by the vector function \( \mathbf{r}(t) = < t - 2, t^2 + 1 > \).

1. What is the position of the object at time \( t = 6 \)?

2. At what time is the object at position \((1, 10)\)?

3. Does the object pass through the point \((7, 50)\)?

4. Find an equation in \( x \) and \( y \) whose graph is the path of the object and sketch the graph.
Vector Equation of a Line: If \( P_0(x_0, y_0) \) is a point on the line with position vector \( \mathbf{r}_0 \) and \( \mathbf{v} \) is a vector parallel to a line, then the vector equation of the line is \( \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v} \).

Parametric equations of the line that passes through the point \( P(x_0, y_0) \) and is parallel to the vector \( <a, b> \) are given by

\[
x = x_0 + at, \quad y = y_0 + bt
\]

Example: Find a vector equation of the line that passes through the point \((-1, 3)\) and is parallel to the vector \( \mathbf{a} = <5, 6> \). What is the slope of this line?

Example: Find parametric equations for the line that passes through the point \((-1, 3)\) and is perpendicular to the vector \( \mathbf{a} = <5, 6> \). What is the slope of this line?
Example: Find a vector equation and parametric equations for the line that passes through the points \((2, 5)\) and \((-1, 7)\).

Determine whether the lines \(r_1(t) = (-1 - 2t)i + (2 + t)j\) and \(r_2(s) = (5 + 3s)i + (3 + 6s)j\) are parallel, perpendicular, or neither. If they are not parallel, find the point of intersection.