1.1 Vectors

A vector is a quantity that has both magnitude and direction. Vectors are drawn as directed line segments and are denoted by boldface letters $\mathbf{a}$ or by $\vec{a}$.

The magnitude of a vector $\mathbf{a}$ is its length, denoted $|\mathbf{a}|$.

In practice, a two-dimensional vector is an ordered pair $\mathbf{a} = <a_1, a_2>$ of real numbers. The numbers $a_1$ and $a_2$ are called the components of $\mathbf{a}$.

Two vectors are equal if they have the same magnitude and direction. So, it doesn’t matter where the vector is as long as it has the same magnitude and direction (the same displacement).

A vector $<a_1, a_2>$ with initial point at the origin is called the position vector of the point $(a_1, a_2)$. A vector with initial point at the origin is said to be in standard position.

Magnitude: The magnitude (or length) of a vector $\mathbf{a} = <a_1, a_2>$ is denoted $|\mathbf{a}|$ where $|\mathbf{a}| = \sqrt{(a_1)^2 + (a_2)^2}$

Given the points $A(x_1, y_1)$ and $B(x_2, y_2)$, the vector $\mathbf{a}$ with initial point $A$ and terminal point $B$ (also written $\overrightarrow{AB}$) is

$$\mathbf{a} = <x_2 - x_1, y_2 - y_1>$$

The length of the vector $\overrightarrow{AB}$ from $A(x_1, y_1)$ to $B(x_2, y_2)$ is

Example: Given the points $A(-2, 5)$ and $B(4, 1)$, find the vector $\overrightarrow{AB}$ and its magnitude.
The zero vector, $\mathbf{0}$, is the vector $< 0, 0 >$. It has length 0 and no direction.

**Vector Addition:** If $\mathbf{a} = < a_1, a_2 >$ and $\mathbf{b} = < b_1, b_2 >$, then the vector $\mathbf{a} + \mathbf{b}$ is $< a_1 + b_1, a_2 + b_2 >$.

Scalar Multiplication: Multiplying a vector by a scalar (number) changes the magnitude of the vector by this factor. A negative scalar changes the magnitude and also reverses the direction.

If $c$ is a scalar and $\mathbf{a} = < a_1, a_2 >$, then the vector $c \mathbf{a} = < ca_1, ca_2 >$.

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if $\mathbf{b} = c \mathbf{a}$ for some scalar $c$.

**Vector Difference:** If $\mathbf{a} = < a_1, a_2 >$ and $\mathbf{b} = < b_1, b_2 >$, then the vector $\mathbf{a} - \mathbf{b}$ is $< a_1 - b_1, a_2 - b_2 >$.

Example: If $\mathbf{a} = < 3, -4 >$ and $\mathbf{b} = < -1, 2 >$, find $|2\mathbf{a} - \mathbf{b}|$.

For more vector properties, see p. 51 of the textbook.
**Unit Vector:** A vector with length (magnitude) 1 is called a **unit vector**.

A unit vector in the direction of \( \mathbf{a} \) is:

Example: If \( \mathbf{a} = \langle -2, 3 \rangle \), find a unit vector in the direction of \( \mathbf{a} \).

**Basis Vectors:** There are two special unit vectors that we use all the time:

\[
\mathbf{i} = \langle 1, 0 \rangle \quad \mathbf{j} = \langle 0, 1 \rangle
\]

\( \mathbf{i} \) is the unit vector in the positive horizontal (\( x \)) direction, and \( \mathbf{j} \) is the unit vector in the positive vertical (\( y \)) direction.

The vectors \( \mathbf{i} \) and \( \mathbf{j} \) are called **basis vectors** because EVERY vector \( \mathbf{a} = \langle a_1, a_2 \rangle \) can be written in terms of these unit vectors by

\[
\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j}
\]

Show:

**Direction:** The direction of a vector is the positive angle \( \theta \) formed by the positive \( x \)-axis and the vector. Suppose you are given a vector \( \mathbf{a} \) with magnitude \( |\mathbf{a}| \) and direction \( \theta \). Then

\[
\mathbf{a} = |\mathbf{a}| \cos \theta, \quad |\mathbf{a}| \sin \theta
\]
Example: A vector \( \mathbf{a} \) has \( |\mathbf{a}|=4 \) and \( \theta = 60^\circ \). Write \( \mathbf{a} \) in component form.

Example: Find the vector \( \mathbf{a} \) that has \( |\mathbf{a}| = 10 \) and makes an angle of \( 330^\circ \) with the positive \( x \)-axis.

Example: Find the direction of the vector \( \mathbf{a} = -4\mathbf{i} - 5\mathbf{j} \).

Applications
The velocity of an object can be modeled by a vector, where the direction of the vector is the direction of motion, and the magnitude of the vector is the speed.

Often when dealing with boats or planes, directions are expressed as **bearings**:
Example: A person is swimming due north at 5 mi/h relative to the water. The water is flowing due west at 2 mi/h. Find the true course and speed of the person.

Example: A jet is flying through a wind that is blowing with a speed of 40 mph in the direction N 30° W. The jet has an airspeed (speed in still air) of 760 mph, and the pilot heads the jet in the direction N 45° E. Find the ground speed and true course (as a bearing) of the jet.
Another application of vectors involves forces. A force has magnitude (measured in pounds or Newtons) and a direction. If several forces are acting on an object, the **resultant force** is the vector sum of all these forces.

Example: Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are acting on an object at a point $P$. $\mathbf{F}_1$ has a magnitude of 40 lbs in a direction of $60^\circ$ from the positive $x$-axis. $\mathbf{F}_2$ has a magnitude of 20 lbs in a direction of $330^\circ$ from the positive $x$-axis. Find the resultant force $\mathbf{F}$ along with both its magnitude and direction.

Example: Consider the triangle $ABC$ shown below where $P$ and $Q$ are the midpoints of $\overline{AB}$ and $\overline{BC}$ respectively. Show that $\overrightarrow{PQ} = \frac{1}{2} \overrightarrow{AC}$.