2.1 The Multiplication Principle and Permutations

Suppose a task $T_1$ can be performed in $N_1$ ways, a task $T_2$ can be performed in $N_2$ ways, ..., and a task $T_n$ can be performed in $N_n$ ways. Then, the number of ways of performing all the tasks $T_1, T_2, \ldots, T_n$ in succession is $N_1 N_2 \cdots N_n$.

This means that the total number of ways to do a series of tasks is the PRODUCT of the number of ways to do each task.

Examples

1. At a restaurant, there are 5 appetizers on the menu, 10 main entrees, and 4 desserts. How many ways can a meal be ordered if one appetizer, one main entree, and one dessert are chosen?

2. Suppose I toss a coin 3 times and record the sequence of heads and tails and then roll a fair 6-sided die 4 times and record the number rolled. How many different outcomes are possible?

3. A test is given with 5 True/False questions and 20 multiple choice questions, each with 5 possible answers. If a True/False question can be left blank, but all MC questions must be answered, how many ways are there to complete the test?

4. How many 7-digit numerical codes are possible if

   (a) there are no restrictions?

   (b) the first digit cannot be 0?

   (c) no repetitions are allowed?

   (d) the first digit cannot be 0 and no repetitions are allowed?

   (e) the first digit cannot be 0, no repetitions are allowed, and the last digit must be odd?
5. How many 4-letter “words” are possible in the English language (assuming we don’t care if the word makes sense) if

(a) there are no restrictions?

(b) repetitions are not allowed?

(c) repetitions are not allowed and the word must start with and end with a vowel?

I have 10 CDs and I want to put them in my CD case which has room for 10 CDs. In how many ways can I do this?

Suppose I have a CD case which holds only 6 CDs. Then how many ways are there to put my 10 CDs in the case?

These two above are examples of permutations. A permutation of a set is an ordered arrangement of the objects in the set. With permutations, ORDER MATTERS.

In the first example, we found the number of ways to arrange all 10 of the 10 CDs by computing $10 \cdot 9 \cdot 8 \cdots 2 \cdot 1$. This is called a factorial.

Factorial: $n! = n(n-1)(n-2)\cdots3 \cdot 2 \cdot 1$ [The factorial is found by pressing MATH and scrolling to PRB.]

So, the answer to the first question can be written and computed as $10!$.

In general, the number of ways to arrange all $n$ objects from a set of $n$ distinct objects is $n!$.

If we arrange only a subset of the $n$ objects, as in the second example above, we get a smaller number. The number of permutations of $r$ objects taken from a set of $n$ distinct objects is denoted $P(n, r)$.

$$P(n, r) = \frac{n!}{(n-r)!}$$

You don’t have to memorize this formula as we can calculate the number of permutations on the calculator. Go to MATH, then PRB, then 2:nPr. On the home screen, type: $n \ nPr \ r$

For the second example above, we could have also found the answer by computing $P(10, 6)$.

$P(n, n)$, which represents the number of ways to arrange all $n$ of $n$ distinct objects is the same as ________.
Example: In how many ways can the letters in the word COPIER be arranged? In how many ways can three of the letters be arranged?

Example: In how many ways can gold, silver, and bronze medals be given in an Olympic track race if there are 10 people running the race?

Example: A faculty advisor is going to choose a president, vice president, secretary, and treasurer for a student organization from a pool of 9 candidates. In how many ways can this be done?

Example: I have 3 different math books, 5 different history books, and 4 different science books that I want to put on my bookshelf. In how many ways can all the books be arranged on the shelf?

In how many ways can 10 of the books be arranged on the shelf?

In how many ways can all of the books be arranged on the shelf if I want books of the same subject matter to be stacked together?

Example: Carlos and Cami go to the movies with 6 of their friends.

- In how many ways can all 8 of them be seated in a row?

- In how many ways can they be seated if Carlos and Cami must sit in the middle two seats?

- In how many ways can they be seated if Carlos and Cami must sit together and the other 6 friends must sit together.
With permutations, remember ORDER MATTERS. For example, the word COPIER is different from the word OPCERI. They are different permutations.

Sometimes in a set of objects, there are objects which are not all distinct. For example, suppose I have 5 identical red marbles, 6 identical green marbles, and 3 identical blue marbles and I want to know how many DISTINCTIABLE ways there are to arrange the marbles in a row. If I just said the answer was 14!, what’s the problem?

Suppose you are given a set of \( n \) objects in which \( n_1 \) objects are alike of one kind, \( n_2 \) are alike of another kind, \( \ldots \), and \( n_r \) are alike of another kind so that \( n_1 + n_2 + \cdots + n_r = n \). Then, the number of permutations of these \( n \) objects taken \( n \) at a time is

\[
\frac{n!}{n_1!n_2!\cdots n_r!}
\]

Finish the marble problem.

Example: How many ways are there to arrange the letters of the word CINCINNATI?

2.2 Combinations

If order does NOT matter, then we are finding a combination instead of a permutation.

A combination is a selection of objects from a set, in which the ORDER DOES NOT MATTER.

The number of combinations of \( n \) distinct objects taken \( r \) at a time is given by

\[
C(n, r) = \frac{n!}{r!(n-r)!}
\]

You don’t have to memorize this. To do combinations on your calculator, go to MATH, then PRB, then \( \wedge nCr \). On the home screen type: \( n \) \( nCr \) \( r \)

Remember, \( n \) is how many there are total of what you want and \( r \) is how many you actually want to choose.

Example: How many ways are there to deal a 5-card hand out of a standard deck of 52 cards?
Example: There are 8 seniors and 6 juniors in the Math Club at a high school. In how many ways can a math team consisting of 4 seniors and 2 juniors be selected from the members of the Math Club?

Example: A business has 5 employee vacancies to fill. In how many ways can the business fill these 5 positions from a group of ten female and ten male applicants if the positions

- May be filled by any combination of men and women.
- Must be filled by 2 men and 3 women?

When you are given a counting problem, the first question to ask yourself is DOES ORDER MATTER? This will tell you whether to use permutations or combinations. Sometimes, you may have to use both of these along with the multiplication principle.

If you need to find the number of ways to do $A$ OR $B$, you have to be careful about double counting. You must remember to use the union rule.

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

More Examples:

1. In how many ways can three 5-card poker hands be dealt out of a standard deck?

2. In how many ways can a committee consisting of a president, a treasurer, a secretary, and 6 regular members be selected from a group of 15 people?
3. At a speech competition, the judges will award one superior ribbon, one excellent ribbon, 3 good ribbons, and 4 honorable mention ribbons. If there are 20 students in the competition, in how many ways can the awards be given?

4. From a shipment of 60 transistors, 5 of which are defective, a sample of 4 transistors is selected at random.
   
   (a) In how many different ways can the sample be selected?

   (b) How many samples contain exactly 3 defective transistors?

   (c) How many samples do not contain any defective transistors?

   (d) How many samples contain exactly 2 or exactly 3 defectives?

   (e) How many samples contain at least 1 defective transistor?
5. A sample of 5 marbles is to be selected at random from a jar containing 15 marbles. If 6 marbles are green, 5 marbles are white, and 4 marbles are black,

(a) How many different samples can be selected?

(b) How many samples contain exactly 2 green and 1 black?

(c) How many samples contain exactly 2 green or exactly 3 green?

(d) How many samples contain exactly 2 green or exactly 1 white?

(e) How many samples contain at least 2 black marbles?

(f) How many samples contain at most 4 green marbles?
2.3 Probability Applications of Counting Principles

Recall from Chapter 1, that for an event \( E \), \( P(E) = \frac{n(E)}{n(S)} \).

We are just adding one extra step to the previous two sections. We don’t want just the number of ways for something to happen, we want the probability it happens, so we must divide by the total number of possibilities.

1. Three cards are selected at random from a standard deck of cards. Find the probability that

   (a) Exactly two spades are drawn.

   (b) All three cards are the same suit.

   (c) A 3-of-a-kind is drawn.

   (d) At least 1 Ace is drawn.
2. A desk contains 4 paper clips, 6 rubber bands, and 3 erasers. Suppose I pick 5 objects at random from the desk. Find the probability that

(a) Exactly 3 of the same item are selected?

(b) Exactly 2 are paper clips or exactly 2 are rubber bands.

3. A computer store contains 80 colored ink cartridges for an ink-jet printer. 6 of the cartridges are defective. If a customer selects 4 of these cartridges at random from the shelf, what is the probability that no more than 1 of them is defective?

4. A candy store has 100 jolly ranchers, of which 20 are green apple. What is the probability that a bag of 10 jolly ranchers contains at least 2 green apples?